O. ARYASOVA AND A. PILIPENKO

ON EXPONENTIAL DECAY OF A DISTANCE BETWEEN SOLUTIONS OF AN SDE WITH NON-REGULAR DRIFT

To the memory of our colleague Sergey Makhno

We consider a multidimensional stochastic differential equation with a Gaussian noise and a drift vector having a jump discontinuity along a hyperplane. The large time behavior of the distance between two solutions starting from different points is studied. We find a sufficient condition for the exponential decay of the distance if the drift does not satisfy a dissipative condition on a given hyperplane.

1. INTRODUCTION

Consider a d-dimensional stochastic differential equation (SDE)

$$
(1) \qquad
\begin{cases}
d\varphi_t(x) = \left(-\lambda\varphi_t(x) + \alpha(\varphi_t(x))\right) dt + \sum_{k=1}^{m} \sigma_k(\varphi_t(x))dw_k(t), t \geq 0, \\
\varphi_0(x) = x,
\end{cases}
$$

where $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$, $\lambda > 0$, $(w(t))_{t\geq 0} = (w_1(t), \ldots, w_m(t))_{t\geq 0}$ is a standard m-dimensional Wiener process, $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma = (\sigma_1, \ldots, \sigma_m) : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ are measurable functions.

It is well known that if α, σ_k are Lipschitz continuous and λ is large enough, then the distance between solutions $\varphi_t(x_1)$ and $\varphi_t(x_2)$ to (1), which starting from two different points x_1 and x_2, converges to 0 in $L_p(\Omega, \mathcal{F}, \mathbb{P})$ as $t \to \infty$, (e.g., [8], [12]). Moreover, the solutions converge themselves to a stationary solution of (1). Lipschitz continuity of α may be relaxed; it can be replaced, for example, by dissipative assumption. It worth noting a recent work by Flandoli et al. [6]. The authors consider an SDE with drift belonging to C^1 and being dissipative out of some bounded set U and a diffusion coefficient σ being constant. They prove the synchronization of the flow for large enough σ. Despite the drift is not supposed to be globally dissipative, the technique of their paper does not allow to consider a discontinuous drift.

In one-dimensional case, results on exponential decay of a distance between solutions to an SDE with non-regular drift were obtained in [2, 3].

We discuss a similar problem if σ_k are Lipschitz continuous but α may have a jump discontinuity at a hyperplane. We do not assume that coefficients of the equation satisfy dissipative conditions, and the results on convergence of solutions to zero are new. As a corollary of our results on convergence of distance between solutions, we get existence and uniqueness of a stationary solution of

$$
\varphi_t = \varphi_s + \int_s^t (-\lambda\varphi_u + \alpha(\varphi_u))du + \sum_{k=1}^{m} \int_s^t \sigma_k(\varphi_u)dw_k(u), \ s \leq t.
$$

Note that all our results concern strong solutions to SDEs, i.e., all solutions are defined on the given probability space and expectations are taken with respect to the given probability measure. If one is interested in weak solutions and a distance between distributions of $\varphi_t(x)$ and $\varphi_t(y)$, then assumptions on coefficients may be relaxed essentially, see for example [10, 11].

2. Large time behavior of the distance between two solutions

We consider the SDE (1). Denote

$$S = \{x \in \mathbb{R}^d : x^d = 0\},$$

$$\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^d > 0\}, \mathbb{R}^d_- = \{x \in \mathbb{R}^d : x^d < 0\}.$$

In what follows we assume that coefficients of (1) satisfy the following conditions.

(A1) The function α is bounded.

(A2) *Lipschitz continuity on* $\mathbb{R}^d_\pm$: There exists $\tilde{K}_\alpha > 0$ such that for all $x, y \in \mathbb{R}^d_+$ or $x, y \in \mathbb{R}^d_-$,

$$|\alpha(x) - \alpha(y)| \leq \tilde{K}_\alpha |x - y|.$$

It follows from (A2) that for all $\tilde{x} \in S$, there exist limits

$$\alpha_+(\tilde{x}) := \lim_{\substack{x \to \tilde{x}, \\ x \in \mathbb{R}^d_+}} \alpha(x), \alpha_-(\tilde{x}) := \lim_{\substack{x \to \tilde{x}, \\ x \in \mathbb{R}^d_-}} \alpha(x).$$

(B1) The function σ is bounded.

(B2) *Lipschitz continuity on* $\mathbb{R}^d$: There exists $\tilde{K}_\sigma > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$|\sigma(x) - \sigma(y)| \leq \tilde{K}_\sigma |x - y|.$$

(B3) *Uniform ellipticity*: There exists a constant $B_\sigma > 0$ such that for all $x \in \mathbb{R}^d$, $\theta \in \mathbb{R}^d$,

$$(2) \qquad\qquad \theta^* \sigma(x) \sigma^*(x) \theta \geq B_\sigma |\theta|^2.$$

Under these assumptions there exists a unique strong solution to (1) (see, for example, [16]).

Remark 2.1. Note that since σ is uniformly elliptic, the solution to equation (1) spends zero time on S. So we can redefine the function α on S in an arbitrary way.

The main result of the paper is following:

Theorem 2.1. *Let conditions (A1), (A2), (B1), (B2), (B3) hold. Then for any* $p \geq 1$:

$$\exists \Lambda = \Lambda(\alpha, \sigma) > 0 \ \forall \lambda > \Lambda \ \exists C_1 = C_1(\lambda, \alpha, \sigma) > 0 \ \exists C_2 = C_2(\lambda, \alpha, \sigma) > 0 : \ \forall x, y \in \mathbb{R}^d,$$

$$(3) \qquad\qquad (\mathbb{E}|\varphi_t(y) - \varphi_t(x)|^p)^{\frac{1}{p}} \leq C_1 e^{-C_2 t} |y - x|.$$

Here $\varphi_t(x)$ *is a solution to equation (1)* *starting at the point* x.

Remark 2.2. The values of Λ, C_1, C_2 will be defined in the proof.

Proof. It can be checked (see [1], Theorem 4 and (51)) that $\varphi_t(\cdot)$ is Gâteaux differentiable in $L_p(\Omega, \mathcal{F}, \mathbb{P})$, $p > 0$, i.e., for all $x \in \mathbb{R}^d$ there exists $\nabla \varphi_t(x)$ such that for all $v \in \mathbb{R}^d$, $t \geq 0$,

$$(4) \qquad\qquad \mathbb{E} \left| \frac{\varphi_t(x + \varepsilon v) - \varphi_t(x)}{\varepsilon} - \nabla \varphi_t(x) v \right|^p \to 0, \ \varepsilon \to 0.$$

Moreover, the process $(\nabla\varphi_t(x))_{t\geq 0}$ is continuous in x in $L_p(\Omega, \mathcal{F}, \mathbb{P})$, $p > 0$. This imply that for $x, y \in \mathbb{R}^d$,

$$(5) \qquad \varphi_t(y) - \varphi_t(x) = \int_0^1 \big(\nabla\varphi_t(x + \xi(y - x)), y - x\big)d\xi,$$

where $\nabla\varphi_t(\cdot)$ is the derivative.

Using (5) we get

$$(6) \quad \mathbb{E}|\varphi_t(y) - \varphi_t(x)|^p \leq \mathbb{E}\big(\int_0^1 \big|(\nabla\varphi_t(x + \xi(y-x)), y-x\big|d\xi\big)^p \leq |y-x|^p \cdot \sup_{z \in \mathbb{R}^d} \mathbb{E}|\nabla\varphi_t(z)|^p.$$

So to obtain (3) we need to get an estimate for $\sup_{z \in \mathbb{R}^d} \mathbb{E}|\nabla\varphi_t(z)|^p$. We consider the case $p = 1$ only. The general case can be considered similarly.

If $\alpha_+(x) = \alpha_-(x), x \in S$, then $Y_t(x) := \nabla\varphi_t(x)$ is a solution to the SDE

$$(7) \qquad \begin{cases} dY_t(x) = [-\lambda + \nabla\alpha(\varphi_t(x))]Y_t(x)dt + \displaystyle\sum_{k=1}^m \nabla\sigma_k(\varphi_t(x))Y_t(x)dw_k(t), t \geq 0, \\[2mm] Y_0(x) = E, \end{cases}$$

where E is a $d \times d$-identity matrix. This formula is well known when $\alpha, \sigma \in C^1(\mathbb{R}^d)$. For Lipschitz continuous functions α and σ the result can be found in [4], Th. 3.3.1.

Remark 2.3. It follows from Rademacher's theorem that the Lipschitz continuous functions α and σ are differentiable almost everywhere with respect to the Lebesgue measure. We define $\nabla\alpha(\varphi_t(x))$, $\nabla\sigma(\varphi_t(x))$ in an arbitrary way at the points where they do not exist. Since σ is non-degenerate, the distribution of $\varphi_t(x)$ is absolutely continuous. So $\nabla\alpha(\varphi_t(x))$, $\nabla\sigma(\varphi_t(x))$ are defined uniquely up to the set of probability zero.

If $\alpha_+(x) \neq \alpha_-(x), x \in S$, then the distributional derivative of α is equal to

$$\nabla\alpha(x) + D(x)\delta_S, x \in \mathbb{R}^d.$$

Here δ_S is the standard surface measure on S (if $d = 1$, $\delta_S(x)$ is the Dirac delta function), and

$$D(x) = \begin{pmatrix} 0 & \cdots & 0 & \alpha_+^1(x) - \alpha_-^1(x) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_+^d(x) - \alpha_-^d(x) \end{pmatrix}, \ x \in S.$$

Formally, in this case the integral form of equation (7) becomes

$$(8) \quad Y_t(x) = E + \int_0^t [-\lambda + \nabla\alpha(\varphi_s(x))]Y_s(x)ds +$$

$$\int_0^t D(\varphi_s(x))Y_s(x)\delta_S(\varphi_s(x))ds + \sum_{k=1}^m \int_0^t \nabla\sigma_k(\varphi_s(x))Y_s(x)dw_k(s).$$

It was proved in [1] that $Y_t(x)$ is a solution to equation (8), where by

$$\int_0^t D(\varphi_s(x))Y_s(x)\delta_S(\varphi_s(x))ds$$

we mean the integral with respect to the local time of the process $(\varphi_t(x))_{t\geq 0}$ on the hyperplane S:

$$\int_0^t D(\varphi_s(x))Y_s(x)dL_s^S(\varphi(x)),$$

where

$$L_t^S(\varphi(x)) := \operatorname*{l.i.m.}_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{|\langle\varphi_s(x), e_d\rangle|\leq\varepsilon}ds, \ e_d = (0, 0, \ldots, 0, 1).$$

Note that the local time of the process $(\varphi_t(x))_{t\geq 0}$ on the hyperplane S coincides with the local time of the d-th coordinate of the process $(\varphi_t(x))_{t\geq 0}$ at the point 0, which is defined by the formula

$$(9) \qquad L_t^0(\varphi^d(x)) = \underset{\varepsilon\downarrow 0}{\text{l.i.m.}}\,\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{|\varphi_s^d(x)|\leq\varepsilon}ds.$$

Then equation (8) can be rewritten as follows

$$(10) \quad Y_t(x) = E + \int_0^t [-\lambda + \nabla\alpha(\varphi_s(x))]Y_s(x)ds +$$

$$\int_0^t D(\varphi_s(x))Y_s(x)dL_s^0(\varphi^d(x)) + \sum_{k=1}^m \int_0^t \nabla\sigma_k(\varphi_s(x))Y_s(x)dw_k(s).$$

It is known that there exists a unique strong solution to equation (10) (see, for example, [13], Ch. V, Th. 7).

Set

$$(11) \qquad K_\alpha := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}|\nabla\alpha(x)|,$$

$$(12) \qquad K_\sigma := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}|\nabla\sigma(x)|.$$

Here and below we denote by $|\cdot|$ both the Euclidean norm of vectors and the Hilbert-Schmidt norm of matrices. Note that (A2), (B2) are satisfied with $\tilde{K}_\alpha = K_\alpha$, $\tilde{K}_\sigma = K_\sigma$, respectively. Put

$$\|D\|_\infty = \sup_{x\in S}|D(x)|.$$

Define

$$h(t) = (2\lambda - 2K_\alpha - K_\sigma^2)t - 2\|D\|_\infty L_t^0(\varphi^d(x)).$$

Lemma 2.1. *For all $T > 0$,*

$$\sup_{t\in[0,T]}\mathbb{E}e^{h(t)}|Y_t(x)|^2 \leq d.$$

The proof of Lemma follows from Itô's formula. For details, see Appendix.

Using the Hölder inequality we obtain

$$(13) \quad \mathbb{E}|Y_t(x)| \leq \left(\mathbb{E}e^{h(t)}|Y_t(x)|^2\right)^{1/2}\left(\mathbb{E}e^{-h(t)}\right)^{1/2} \leq$$

$$d^{1/2}e^{\left(-\lambda+K_\alpha+\frac{1}{2}K_\sigma^2\right)t}\left(\mathbb{E}e^{2\|D\|_\infty L_t^0(\varphi^d(x))}\right)^{1/2}.$$

Lemma 2.2. *For each $t > 0$,*

$$(14) \qquad \sup_{x\in\mathbb{R}^d}\mathbb{E}L_t^0(\varphi^d(x)) \leq \frac{\rho(t,\lambda)}{B_\sigma},$$

where

$$\rho(t,\lambda) = \|\alpha^d\|_\infty t + \left(1+\frac{2}{3}\lambda t\right)\sqrt{\left(\frac{\|\alpha^d\|_\infty^2}{2\lambda}+\|\sigma\|_\infty^2\right)t}$$

and $\|\alpha\|_\infty = \sup_{x\in\mathbb{R}^d}|\alpha(x)|$, $\|\sigma\|_\infty = \sup_{x\in\mathbb{R}^d}|\sigma(x)|$, B_σ is the uniform ellipticity constant from equation (2).

To prove the Lemma we use Tanaka's formula. See Appendix for details.

It is well known that $L_t^S(\varphi(x))$ is a W-functional of the Markov process $(\varphi_t(x))_{t \geq 0}$ (see [5], Ch. 6–8 for theory and terminology). Then the following estimates on the moments of $L_t^S(\varphi(x))$ are true.

Proposition 2.1 ([7], Ch. II, §6, Lemma 3)**.** *For all $n \geq 1$, $t > 0$,*

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\left(L_t^S(\varphi(x))\right)^n \leq n! \left(\sup_{x \in \mathbb{R}^d} \mathbb{E}L_t^S(\varphi(x))\right)^n.$$

Since $L_t^0(\varphi^d(x)) = L_t^S(\varphi(x))$, then using Proposition 2.1 and Lemma 2.2 we obtain the following modification of Khas'minskii's Lemma (see [9] or [15], Ch.1 Lemma 2.1).

Lemma 2.3. *Let $t_0 > 0$ be such that $\frac{2\|D\|_\infty \rho(t_0, \lambda)}{B_\sigma} < 1$ and (14) hold. Then for all $t \leq t_0$,*

$$(15) \qquad \sup_{x \in \mathbb{R}^d} \mathbb{E}e^{2\|D\|_\infty L_t^0(\varphi^d(x))} \leq \frac{1}{1 - \frac{2\|D\|_\infty}{B_\sigma}\rho(t_0, \lambda)}.$$

Using the inequality (15) we can estimate the right-hand side of (13) for small t.

Consider now an arbitrary $t > 0$. Put $n = \left[\frac{t}{t_0}\right] + 1$, and

$$s_0 = 0, s_1 = t_0, \ldots, s_k = kt_0, \ldots, s_{n-1} = (n-1)t_0, s_n = t.$$

We have

$$\mathbb{E}e^{2\|D\|_\infty L_t^0(\varphi^d(x))} = \mathbb{E}\prod_{k=0}^{n-1} e^{2\|D\|_\infty\left(L_{s_{k+1}}^0(\varphi^d(x)) - L_{s_k}^0(\varphi^d(x))\right)} =$$

$$\mathbb{E}\left(\mathbb{E}\left[\prod_{k=0}^{n-1} e^{2\|D\|_\infty\left(L_{s_{k+1}}^0(\varphi^d(x)) - L_{s_k}^0(\varphi^d(x))\right)}\Big|\mathcal{F}_{s_{n-1}}\right]\right) =$$

$$\mathbb{E}\left\{\prod_{k=0}^{n-2} e^{2\|D\|_\infty\left(L_{s_{k+1}}^0(\varphi^d(x)) - L_{s_k}^0(\varphi^d(x))\right)} \mathbb{E}\left(e^{2\|D\|_\infty\left(L_{s_n}^0(\varphi^d(x)) - L_{s_{n-1}}^0(\varphi^d(x))\right)}\Big|\mathcal{F}_{s_{n-1}}\right)\right\}.$$

It is not hard to see that

$$(16) \qquad P\{L_{t+s}^S(\varphi(x)) = L_s^S(\varphi(x)) + \theta_s L_t^S(\varphi(x)), \ s \geq 0, t \geq 0\} = 1,$$

where θ is the shift operator.

Using (16) and Lemma 2.3 we get for $k = 1, \ldots, n$,

$$\mathbb{E}\left(e^{2\|D\|_\infty\left(L_{s_k}^0(\varphi^d(x)) - L_{s_{k-1}}^0(\varphi^d(x))\right)}\Big|\mathcal{F}_{s_{k-1}}\right) = \mathbb{E}\left(e^{2\|D\|_\infty \theta_{s_{k-1}} L_{s_k - s_{k-1}}^0(\varphi^d(x))}\Big|\mathcal{F}_{s_{k-1}}\right) \leq$$

$$\sup_{z \in \mathbb{R}^d} \mathbb{E}e^{2\|D\|_\infty L_{s_k - s_{k-1}}^0(\varphi^d(z))} \leq \sup_{z \in \mathbb{R}^d} \mathbb{E}e^{2\|D\|_\infty L_{t_0}^0(\varphi^d(z))} \leq \frac{1}{1 - \frac{2\|D\|_\infty}{B_\sigma}\rho(t_0, \lambda)} \ \text{a.s.}$$

Then

$$(17)$$

$$\mathbb{E}e^{2\|D\|_\infty L_t^0(\varphi^d(x))} \leq \frac{1}{1 - \frac{2\|D\|_\infty}{B_\sigma}\rho(t_0, \lambda)} \mathbb{E}\prod_{k=0}^{n-2} e^{2\|D\|_\infty\left(L_{s_{k+1}}^0(\varphi^d(x)) - L_{s_k}^0(\varphi^d(x))\right)} \leq \cdots \leq$$

$$\frac{1}{\left(1 - \frac{2\|D\|_\infty}{B_\sigma}\rho(t_0, \lambda)\right)^n} = e^{-n \ln\left(1 - \frac{2\|D\|_\infty}{B_\sigma}\rho(t_0, \lambda)\right)} = e^{-\left(\left[\frac{t}{t_0}\right] + 1\right)\ln\left(1 - \frac{2\|D\|_\infty}{B_\sigma}\rho(t_0, \lambda)\right)}.$$

The right-hand side of (17) does not depend on x. So we have

$$(18) \qquad \sup_{x \in \mathbb{R}^d} \mathbb{E} e^{2\|D\|_\infty L_t^0(\varphi^d(x))} \le e^{-\left(\left[\frac{t}{t_0}\right]+1\right)\ln\left(1-\frac{2\|D\|_\infty}{B_\sigma}\rho(t_0,\lambda)\right)}.$$

Substituting this inequality into (13) we get the following inequality for any $t > 0$:

$$(19) \quad \sup_{x \in \mathbb{R}^d} \mathbb{E}|Y_t(x)| \le d^{1/2} e^{\left(-\lambda+K_\alpha+\frac{1}{2}K_\sigma^2\right)t} \sup_{x \in \mathbb{R}^d} \left(\mathbb{E} e^{2\|D\|_\infty L_t^0(\varphi^d(x))}\right)^{1/2} \le$$

$$d^{1/2} e^{\left(-\lambda+K_\alpha+\frac{1}{2}K_\sigma^2\right)t-\frac{1}{2}\left(\frac{t}{t_0}+1\right)\ln\left(1-\frac{2\|D\|_\infty}{B_\sigma}\rho(t_0,\lambda)\right)} =$$

$$d^{1/2} \frac{1}{\sqrt{1-\frac{2\|D\|_\infty}{B_\sigma}\rho(t_0,\lambda)}} e^{\left(-\lambda+K_\alpha+\frac{1}{2}K_\sigma^2-\frac{1}{2t_0}\ln\left(1-\frac{2\|D\|_\infty}{B_\sigma}\rho(t_0,\lambda)\right)\right)t}.$$

First, assume that $\Lambda \ge 1/2$. Then for all $\lambda > \Lambda$ and $t \ge 0$,

$$\rho(t,\lambda) \le \left(1+\frac{2}{3}\lambda t\right)\sqrt{\left(\|\alpha^d\|_\infty^2+\|\sigma\|_\infty^2\right)t}+\|\alpha^d\|_\infty t.$$

From (19) we obtain

$$(20) \qquad \sup_{x \in \mathbb{R}^d} \mathbb{E}|Y_t(x)| \le C_1(\lambda,\alpha,\sigma) e^{\left(-\lambda+K-\frac{1}{2t_0}\ln\left(1-K_1 t_0^{1/2}-K_2 t_0-K_3\lambda t_0^{3/2}\right)\right)t},$$

where

$$(21) \qquad C_1(\lambda,\alpha,\sigma) = d^{1/2} \frac{1}{\sqrt{1-\frac{2\|D\|_\infty}{B_\sigma}\rho(t_0,\lambda)}},$$

$$K = K_\alpha + \frac{1}{2}K_\sigma^2,$$

$$K_1 = \frac{2\|D\|_\infty}{B_\sigma}\sqrt{\left(\|\alpha^d\|_\infty^2+\|\sigma\|_\infty^2\right)}; \quad K_2 = \frac{2\|D\|_\infty}{B_\sigma}\|\alpha^d\|_\infty;$$

$$K_3 = \frac{4\|D\|_\infty}{3B_\sigma}\sqrt{\left(\|\alpha^d\|_\infty^2+\|\sigma\|_\infty^2\right)}.$$

If we show that

$$\exists \Lambda \ge 1/2 \; \forall \lambda > \Lambda \; \exists t_0 = t_0(\lambda) > 0 : 0 \quad 2t_0(-\lambda+K)-\ln(1-K_1\lambda t_0^{3/2}-K_2 t_0-K_3 t_0^{1/2}) < 0,$$

then (3) will follow from (6) and (20).

Note that for all $\lambda \ge \frac{4}{3}K$,

$$(22) \qquad 2t_0(-\lambda+K) \le -\frac{\lambda t_0}{2}.$$

Further, it is easy to see that there exist $\delta > 0$ such that

$$(23) \qquad -\frac{1}{2}-\ln\left(1-K_2\delta-(K_1+K_3)\delta^{1/2}\right) < 0.$$

Put $\Lambda = \max\left\{\frac{1}{2}, \frac{4}{3}K, \frac{1}{\delta}\right\}$. Now for each $\lambda > \Lambda$ we can choose $t_0 = t_0(\lambda) > 0$, which satisfies conditions of Lemma 2.3 and such that $t_0 < \frac{1}{\lambda}$. In particular, this implies that

$t_0 < \delta$. Then using (22), (23) we get

$$2t_0(-\lambda + K) - \ln(1 - K_1\lambda t_0^{3/2} - K_2t_0 - K_3t_0^{1/2}) \leq$$

$$-\frac{\lambda t_0}{2} - \ln(1 - K_1(\lambda t_0)t_0^{1/2} - K_2t_0 - K_3t_0^{1/2}) \leq$$

$$-\frac{1}{2} - \ln(1 - K_1t_0^{1/2} - K_2t_0 - K_3t_0^{1/2}) \leq$$

$$-\frac{1}{2} - \ln\left(1 - K_2\delta - (K_1 + K_3)\delta^{1/2}\right) < 0.$$

Hence, if $\lambda > \Lambda$, t_0 satisfies the conditions of Lemma 2.3, and $t_0 < \frac{1}{\lambda}$, then there exist $C_1 = C_1(\lambda, \alpha, \sigma) > 0$ defined by (21) and

$$C_2 = C_2(\lambda, \alpha, \sigma) = 2t_0(\lambda - K) + \ln(1 - K_1\lambda t_0^{3/2} - K_2t_0 - K_3t_0^{1/2}) > 0$$

such that the inequality

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}|Y_t(x)| \leq C_1 e^{-C_2 t}$$

holds.

Similarly we can get the estimate

(24) $$\sup_{x \in \mathbb{R}^d} \mathbb{E}|Y_t(x)|^p \leq C_1(p)e^{-C_2(p)t}$$

for any $p \geq 1$.

Substituting (24) into (6) we get (3). □

3. STATIONARY SOLUTION

Let $(\tilde{w}_1(t), \ldots, \tilde{w}_m(t))_{t \geq 0}$ and $(\hat{w}_1(t), \ldots, \hat{w}_m(t))_{t \geq 0}$ be standard independent m-dimensional Wiener processes. For $1 \leq k \leq m$ define two-sided Brownian motions:

$$w_k(t) = \begin{cases} \tilde{w}_k(t), t \geq 0, \\ \hat{w}_k(-t), t < 0. \end{cases}$$

Let $\mathcal{F}_t$ be the augmentation of σ-algebra generated by $\{w_k(s), \ s \leq t, \ 1 \leq k \leq m\}$.

Consider a d-dimensional SDE

(25) $$d\varphi_t = (-\lambda\varphi_t + \alpha(\varphi_t)) \, dt + \sum_{k=1}^m \sigma_k(\varphi_t)dw_k(t), \ t \in \mathbb{R},$$

where λ, α, σ satisfy the conditions of Theorem 2.1.

Definition 3.1. We say that $\mathcal{F}_t$-adapted continuous process $(\varphi_t)_{t \in \mathbb{R}}$ is a stationary solution to equation (25) if for all $s, t \in \mathbb{R}$ such that $s \leq t$,

$$\varphi_t = \varphi_s + \int_s^t (-\lambda\varphi_u + \alpha(\varphi_u)) \, du + \sum_{k=1}^m \int_s^t \sigma_k(\varphi_u)dw_k(u) \text{ a.s.,}$$

and the process $(\varphi_t)_{t \in \mathbb{R}}$ is strictly stationary.

Theorem 3.1. *Let λ, α, σ satisfy the conditions of Theorem 2.1. Then there exists a unique stationary solution to equation (25).*

Proof. The proof of this theorem is quite standard, see [12]. So we outline only the main steps without technical details. *Existence.* Denote by $\varphi_{s,t}(x)$, $t \in [s, \infty)$, a solution to the SDE

(26) $$\begin{cases} d\varphi_{s,t}(x) = [-\lambda\varphi_{s,t}(x) + \alpha(\varphi_{s,t}(x))] \, dt + \sum_{k=1}^m \sigma_k(\varphi_{s,t}(x))dw_k(t), \ t \geq s, \\ \varphi_{s,s}(x) = x. \end{cases}$$

A stationary solution is looked as a limit in L_2 of $\varphi_{s,t}(0)$ as $s \to -\infty$.

Lemma 3.1. *For all $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$,*

$$\sup_{t \in [s,\infty)} \mathbb{E}|\varphi_{s,t}(x)|^2 < \infty. \tag{27}$$

Proof. Let $f \in C^2(\mathbb{R}^d)$ and A be the infinitesimal generator of the process $(\varphi_{s,t}(x))_{t \geq 0}$:

$$Af(x) = \sum_{i=1}^d (-\lambda x^i + \alpha^i(x)) \frac{\partial f}{\partial x^i}(x) + \sum_{i,j=1}^d (\sigma(x)\sigma(x)^T)_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(x).$$

It is well known (e.g. [11], §3.2) that to prove (27) it is enough to verify that there exist $K_1, K_2 > 0$ such that for all $x \in \mathbb{R}^d$,

$$A|x|^2 \leq K_1 - K_2|x|^2. \tag{28}$$

We have

$$A|x|^2 = -2\lambda|x|^2 + 2(\alpha(x), x) + |\sigma|^2.$$

It is easy to see that (28) is satisfied with, for example, $K_1 = \frac{\|\alpha\|_\infty^2}{\lambda} + \|\sigma\|_\infty^2$, $K_2 = \lambda$. Recall that $\|\alpha\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |\alpha(x)|$. $\square$

Let $t \in \mathbb{R}$, $s \leq t$. It follows from the uniqueness of the strong solution to (26) that

$$\varphi_{s-p,t}(0) = \varphi_{s,t}(\varphi_{s-p,s}(0)) \text{ a.s.}$$

By Theorem 2.1 and Lemma 3.1,

$$\sup_{p \geq 0} \mathbb{E}|\varphi_{s-p,t}(0) - \varphi_{s,t}(0)| = \sup_{p \geq 0} \mathbb{E}|\varphi_{s,t}(\varphi_{s-p,s}(0)) - \varphi_{s,t}(0)| \leq$$

$$\sup_{p \geq 0} C_1 e^{C_2(s-t)} \mathbb{E}|\varphi_{s-p,s}(0) - 0| \leq C_3 e^{C_2(s-t)} \to 0, \ s \to -\infty.$$

Here C_1, C_2 are constants from Theorem 2.1, C_3 is some positive constant that comes from Lemma 3.1.

Therefore there exists a limit

$$\psi(t) := L_2 \lim_{s \to -\infty} \varphi_{s,t}(0).$$

Stationarity of $\psi(t)$ follows from the construction.

Theorem 2.1 and the construction of $\psi(t)$ yield that for any $s \leq t$:

$$\psi(t) = \varphi_{s,t}(\psi(s)) \text{ a.s.}$$

It follows easily from the last equation that $\psi(t)$ has a continuous modification.

Uniqueness. Let $(\tilde{\psi}(t))_{t \in \mathbb{R}}$ be another stationary solution, possibly without finite moments. We have for any $s \leq t$:

$$\mathbb{E}\left(|\tilde{\psi}(t) - \psi(t)| \wedge 1\right) = \mathbb{E}\left(|\varphi_{s,t}(\tilde{\psi}(s)) - \varphi_{s,t}(\psi(s))| \wedge 1\right) \leq \tag{29}$$

$$\mathbb{E}\left(\mathbb{E}\left(|\varphi_{s,t}(\tilde{\psi}(s)) - \varphi_{s,t}(\psi(s))| \wedge 1\right)|\mathcal{F}_s\right) = \mathbb{E}\left(\mathbb{E}\left(|\varphi_{s,t}(x) - \varphi_{s,t}(y)| \wedge 1\right)\Big|_{\substack{x=\tilde{\psi}(s),\\y=\psi(s)}}\right) \leq$$

$$\mathbb{E}\left((\mathbb{E}|\varphi_{s,t}(x) - \varphi_{s,t}(y)|)\Big|_{\substack{x=\tilde{\psi}(s),\\y=\psi(s)}} \wedge 1\right) \leq \mathbb{E}\left(|x-y|C_1 e^{-C_2(t-s)}\Big|_{\substack{x=\tilde{\psi}(s),\\y=\psi(s)}} \wedge 1\right) =$$

$$\mathbb{E}\left(|\tilde{\psi}(s) - \psi(s)|C_1 e^{-C_2(t-s)} \wedge 1\right) \leq$$

$$\mathbb{E}\left[|\psi(s)|C_1 e^{-C_2(t-s)} \wedge 1\right] + \mathbb{E}\left[|\tilde{\psi}(s)|C_1 e^{-C_2(t-s)} \wedge 1\right] =$$

$$\mathbb{E}\left[|\psi(0)|C_1 e^{-C_2(t-s)} \wedge 1\right] + \mathbb{E}\left[|\tilde{\psi}(0)|C_1 e^{-C_2(t-s)} \wedge 1\right].$$

Here C_1, C_2 are constants from (3). By Lebesgue's dominated convergence theorem, the right-hand side of (29) tends to zero as $s \to -\infty$. Hence $\mathbb{E}|\tilde{\psi}(t) - \psi(t)| \wedge 1 = 0$. This and continuity of $(\tilde{\psi}(t)), (\psi(t))$ yield

$$\mathbb{P}(\tilde{\psi}(t) = \psi(t), \ t \in \mathbb{R}) = 1.$$

$\square$

4. APPENDIX

Proof of Lemma 2.1. Put

$$\tau^N = \inf\{t \geq 0 : t + \int_0^t |Y_s(x)|^2 ds + L_t^0(\varphi^d(x)) \geq N\}.$$

By Itô's formula,

$$e^{h(t \wedge \tau^N)}|Y_{t \wedge \tau^N}(x)|^2 = |Y_0(x)|^2 + \int_0^{t \wedge \tau^N} e^{h(s)}|Y_s(x)|^2(2\lambda - 2K_\alpha - K_\sigma^2)ds -$$

$$2\int_0^{t \wedge \tau^N} e^{h(s)}|Y_s(x)|^2|D(\varphi_s(x))|dL_s^0(\varphi(x)) + 2\int_0^{t \wedge \tau^N} e^{h(s)} \sum_{i,j=1}^d Y_s^{ij}(x)dY_s^{ij}(x)ds +$$

$$\sum_{k=1}^m \sum_{i,j=1}^d \int_0^{t \wedge \tau^N} e^{h(s)} \left(\sum_{r=1}^d \nabla\sigma_k^{ir}(\varphi_s(x))Y_s^{rj}(x)\right)^2 ds =$$

$$|Y_0(x)|^2 + \int_0^{t \wedge \tau^N} e^{h(s)}|Y_s(x)|^2(2\lambda - 2K_\alpha - K_\sigma^2)ds -$$

$$2\int_0^{t \wedge \tau^N} e^{h(s)}|Y_s(x)|^2|D(\varphi_s(x))|dL_s^0(\varphi(x)) -$$

$$2\lambda\int_0^{t \wedge \tau^N} e^{h(s)} \sum_{i,j=1}^d (Y_s^{ij}(x))^2 ds + 2\int_0^{t \wedge \tau^N} e^{h(s)} \sum_{i,j,q=1}^d Y_s^{ij}(x)\nabla\alpha^{iq}(\varphi_s(x))Y_s^{qj}(x)ds +$$

$$2\int_0^{t \wedge \tau^N} e^{h(s)} \sum_{i,j,q=1}^d Y_s^{ij}(x)D^{iq}(\varphi_s(x))Y_s^{qj}(x)dL_s^0(\varphi^d(x)) +$$

$$2\int_0^{t \wedge \tau^N} e^{h(s)} \sum_{k=1}^m \sum_{i,j,q=1}^d Y_s^{ij}(x)\nabla\sigma_k^{iq}(\varphi_s(x))Y_s^{qj}(x)dw_k(s) +$$

$$\sum_{k=1}^m \sum_{i,j=1}^d \int_0^{t \wedge \tau^N} e^{h(s)} \left(\sum_{r=1}^d \nabla\sigma_k^{ir}(\varphi_s(x))Y_s^{rj}(x)\right)^2 ds.$$

Using the Cauchy-Schwarz inequality we get that for each $t > 0$,

$$\left|\sum_{i,j,q=1}^d Y_t^{ij}(x)\nabla\alpha^{iq}(\varphi_t(x))Y_t^{qj}(x)\right| \leq |Y_t(x)|^2|\nabla\alpha(\varphi_t(x))|.$$

Then taking into account Remark 2.1 we obtain

$$(30) \qquad \int_0^{t \wedge \tau^N} \left|\sum_{i,j,q=1}^d Y_s^{ij}(x)\nabla\alpha^{iq}(\varphi_s(x))Y_s^{qj}(x)\right| ds \leq K_\alpha \int_0^{t \wedge \tau^N} |Y_s(x)|^2 ds,$$

where K_α is defined by (11).

Similarly,

$$(31) \quad \int_0^{t\wedge\tau^N} e^{h(s)} \left| \sum_{i,j,q=1}^{d} Y_s^{ij}(x) D^{iq}(\varphi_s(x)) Y_s^{qj}(x) \right| dL_s^0(\varphi^d(x)) \leq$$

$$\|D\|_\infty \int_0^{t\wedge\tau^N} e^{h(s)} |Y_s(x)|^2 dL_s^0(\varphi^d(x)).$$

Further, for $t > 0$,

$$(32) \quad \sum_{k=1}^{m} \sum_{i,j=1}^{d} \left(\sum_{r=1}^{d} \nabla\sigma_k^{ir}(\varphi_t(x)) Y_t^{rj}(x) \right)^2 \leq$$

$$\sum_{k=1}^{m} \sum_{i,j=1}^{d} \left(\sum_{r=1}^{d} \left(\nabla\sigma_k^{ir}(\varphi_t(x)) \right)^2 \right) \left(\sum_{r=1}^{d} \left(Y_t^{rj}(x) \right)^2 \right) \leq |\nabla\sigma(x)|^2 |Y_t(x)|^2 \leq K_\sigma^2 |Y_t(x)|^2,$$

where K_σ is defined by (12).

Taking into account (30)–(32) we get

$$e^{h(t\wedge\tau^N)} |Y_{t\wedge\tau^N}(x)|^2 = |Y_0(x)|^2 + (2\lambda - 2K_\alpha - K_\sigma^2) \int_0^{t\wedge\tau^N} e^{h(s)} |Y_s(x)|^2 ds -$$

$$2 \int_0^{t\wedge\tau^N} e^{h(s)} |Y_s(x)|^2 \|D\|_\infty dL_s^0(\varphi(x)) - 2\lambda \int_0^{t\wedge\tau^N} e^{h(s)} |Y_s(x)|^2 ds +$$

$$2dK_\alpha \int_0^{t\wedge\tau^N} e^{h(s)} |Y_s(x)|^2 ds + 2 \int_0^{t\wedge\tau^N} e^{h(s)} \|D\|_\infty |Y_s(x)|^2 dL_s^0(\varphi^d(x)) +$$

$$K_\sigma^2 \int_0^{t} e^{h(s)} |Y_s(x)|^2 ds +$$

$$2 \int_0^{t\wedge\tau^N} e^{h(s)} \sum_{k=1}^{m} \sum_{i,j,q=1}^{d} Y_s^{ij}(x) \nabla\sigma_k^{iq}(\varphi_s(x)) Y_s^{qj}(x) dw_k(s) \leq |Y_0(x)|^2 + M(t\wedge\tau^N),$$

where

$$M(t\wedge\tau^N) = 2 \int_0^{t\wedge\tau^N} e^{h(s)} \sum_{k=1}^{m} \sum_{i,j,q=1}^{d} Y_s^{ij}(x) \nabla\sigma_k^{iq}(\varphi_s(x)) Y_s^{qj}(x) dw_k(s), t \geq 0,$$

is a square integrable martingale. Then for all $t \geq 0$,

$$\mathbb{E} e^{h(t\wedge\tau^N)} |Y_{t\wedge\tau^N}(x)|^2 \leq |Y_0(x)|^2 = |E|^2 = d.$$

Passing to the limit as $N \to \infty$ and applying Fatou's lemma we get that for all $T > 0$,

$$\sup_{t\in[0,T]} \mathbb{E} e^{h(t)} |Y_t(x)|^2 \leq d.$$

$\square$

Proof of Lemma 2.2. The process $(\varphi_t(x))_{t\geq 0}$ is a multidimensional semimartingale. Set

$$(33) \quad \widetilde{L}_t^0(\varphi^d(x)) = 2(\varphi_t^d(x))^+ - 2(x^d)^+ - 2 \int_0^t \mathbb{1}_{\varphi_s^d(x)>0} d\varphi_s^d(x), \; t \geq 0.$$

This process is also called a local time of the process $(\varphi_t^d(x))_{t\geq 0}$ at zero and satisfies the equality (see [14], Ch. VI, Corollary (1.9) and Th. (1.7))

$$\widetilde{L}_t^0(\varphi^d(x)) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{(-\varepsilon,\varepsilon)}(\varphi_s^d(x))d\langle\varphi^d(x),\varphi^d(x)\rangle_s =$$

$$= \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{(-\varepsilon,\varepsilon)}(\varphi_s^d(x))\sum_{k=1}^d \left(\sigma_k^d(\varphi_s(x))\right)^2 ds,$$

which holds almost surely.

Using (9) and (B3) in which we put $\theta^* = (0,0,\ldots,0,1)$ we get that, almost surely,

$$\widetilde{L}_t^0(\varphi^d(x)) \geq B_\sigma \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{(-\varepsilon,\varepsilon)}(\varphi_s^d(x))ds = B_\sigma L_t^0(\varphi^d(x)) = B_\sigma L_t^0(\varphi(x)).$$

So

(34) $$L_t^0(\varphi^d(x)) \leq \frac{1}{B_\sigma}\widetilde{L}_t^0(\varphi^d(x)).$$

Consequently, to estimate $\mathbb{E}L_t^0(\varphi(x))$ it is enough to get an estimation for $\mathbb{E}\widetilde{L}_t^0(\varphi^d(x))$.

Since the local time $L_t^0(\varphi^d(x)) = L_t^S(\varphi(x))$ does not increase until the first time when the process $(\varphi_t(x))_{t\geq 0}$ reaches the hyperplane S, it follows from the strong Markov property of the process $(\varphi_t(x))_{t\geq 0}$ that

(35) $$\mathbb{E}L_t^0(\varphi^d(x)) \leq \sup_{\tilde{x}\in S}\mathbb{E}L_t^0(\varphi^d(\tilde{x})).$$

To estimate the expectation of the local time let us estimate each term in the right-hand side of Tanaka's formula (33). Note that for all $t \geq 0$,

(36) $$\mathbb{E}(\varphi_t^d(\tilde{x}))^+ \leq \mathbb{E}|\varphi_t^d(\tilde{x})| \leq \sqrt{\mathbb{E}\left(\varphi_t^d(\tilde{x})\right)^2}.$$

By Itô's formula, for any $\tilde{x} \in S$

(37) $$(\varphi_t^d(\tilde{x}))^2 = \varphi_0^d(\tilde{x}))^2 + 2\int_0^t \varphi_s^d(\tilde{x})d\varphi_s^d(\tilde{x}) + \sum_{k=1}^m\int_0^t \left(\sigma_k^d(\varphi_s(\tilde{x}))\right)^2 ds =$$

$$\int_0^t\left[-2\lambda(\varphi_s^d(\tilde{x}))^2 + 2\alpha^d(\varphi_s(\tilde{x}))\varphi_s^d(\tilde{x}) + \sum_{k=1}^m\left(\sigma_k^d(\varphi_s(\tilde{x}))\right)^2\right]ds+$$

$$2\int_0^t \varphi_s^d(\tilde{x})\sum_{k=1}^m \sigma_k^d(\varphi_s(\tilde{x}))dw_k(s).$$

Consider the expression in the square brackets. We have

$$-2\lambda(\varphi_s^d(\tilde{x}))^2 + 2\alpha^d(\varphi_s(\tilde{x}))\varphi_s^d(\tilde{x}) + \sum_{k=1}^m\left(\sigma_k^d(\varphi_s(\tilde{x}))\right)^2 \leq f(\varphi_s^d(\tilde{x})),$$

where $f(x) = -2\lambda x^2 + 2||\alpha^d||_\infty^2 x + ||\sigma||_\infty^2$. The function f attains the global maximum at $x_{max} = \frac{||\alpha^d||_\infty^2}{2\lambda}$, and $f(x_{max}) = \frac{||\alpha^d||_\infty^2}{2\lambda} + ||\sigma||_\infty^2$. Substituting the maximum value of f into (37) we get

$$(\varphi_t^d(\tilde{x}))^2 \leq \left(\frac{||\alpha^d||_\infty^2}{2\lambda} + ||\sigma||_\infty^2\right)t + 2\sum_{k=1}^m\int_0^t \varphi_s^d(\tilde{x})\sigma_k^d(\varphi_s(\tilde{x}))dw_k(s).$$

Here we use the fact that $\varphi_0^d(\tilde{x}) = 0$. Note that $\sum_{k=1}^m\int_0^t \varphi_s^d(\tilde{x})\sigma_k^d(\varphi_s(\tilde{x}))dw_k(s)$ is a local square integrable martingale.

Using localization and Fatou's lemma we obtain

$$\mathbb{E}(\varphi_t^d(\tilde{x}))^2 \leq \left(\frac{||\alpha^d||_\infty^2}{2\lambda} + ||\sigma||_\infty^2\right) t.$$

Hence, by (36)

$$(38) \qquad \mathbb{E}(\varphi_t^d(\tilde{x}))^+ \leq \sqrt{\left(\frac{||\alpha^d||_\infty^2}{2\lambda} + ||\sigma||_\infty^2\right) t}.$$

Further,

$$-\int_0^t \mathbb{1}_{\varphi_s^d(\tilde{x})>0} d\varphi_s^d(\tilde{x}) =$$

$$\lambda \int_0^t \mathbb{1}_{\varphi_s^d(\tilde{x})>0} \varphi_s^d(\tilde{x}) ds - \int_0^t \mathbb{1}_{\varphi_s^d(\tilde{x})>0} \alpha^d(\varphi_s(\tilde{x})) ds - \sum_{k=1}^m \int_0^t \mathbb{1}_{\varphi_s^d(\tilde{x})>0} \sigma_k^d(\varphi_s(\tilde{x})) dw_k(s).$$

Using (38) we get

$$(39) \quad -\mathbb{E}\int_0^t \mathbb{1}_{\varphi_s^d(\tilde{x})>0} d\varphi_s^d(\tilde{x}) \leq ||\alpha^d||_\infty t + \lambda \mathbb{E}\int_0^t (\varphi_s^d(\tilde{x}))^+ ds \leq$$

$$||\alpha^d||_\infty t + \frac{2\lambda}{3}\sqrt{\left(\frac{||\alpha^d||_\infty^2}{2\lambda} + ||\sigma||_\infty^2\right) t^3}.$$

Taking into account (38), (39) we obtain

$$\mathbb{E}\widetilde{L}_t^0(\varphi^d(\tilde{x})) \leq \rho(t,\lambda),$$

where

$$\rho(t,\lambda) = ||\alpha^d||_\infty t + \left(1 + \frac{2}{3}\lambda t\right)\sqrt{\left(\frac{||\alpha^d||_\infty^2}{2\lambda} + ||\sigma||_\infty^2\right) t}.$$

Then (35) implies that for each $x \in \mathbb{R}^d$,

$$\mathbb{E}\widetilde{L}_t^0(\varphi^d(x)) \leq \rho(t,\lambda),$$

and

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\widetilde{L}_t^0(\varphi^d(x)) \leq \rho(t,\lambda).$$

By (34),

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}L_t^0(\varphi^d(x)) \leq \frac{\rho(t,\lambda)}{B_\sigma}.$$

$\square$

REFERENCES

1. O. Aryasova and A. Pilipenko. *A representation for the derivative with respect to the initial data of the solution of an sde with a non-regular drift*, North-W. Eur. J. of Math. **3** (2017), 1–40.
2. O. Aryasova and A. Pilipenko, *Exponential a.s. synchronization of one-dimensional diffusions with non-regular coefficients*, 2020, arXiv2003.02614.
3. O. V. Aryasova and A. Yu. Pilipenko, *On properties of a flow generated by an SDE with discontinuous drift*, Electron. J. Probab. **17** (2012), no. 106, 1–20.
4. N. Bouleau and F. Hirsch, *Dirichlet Forms and Analysis on Wiener Space*, De Gruyter, Berlin, Boston, 2010.

5. E. B. Dynkin, *Markov Processes*, Fizmatlit, Moscow, 1963. [Translated from the Russian to the English by J. Fabius, V. Greenberg, A. Maitra, and G. Majone. Academic Press, New York; Springer, Berlin, 1965. vol. 1, xii + 365 pp.; vol. 2, viii + 274 pp.].

6. F. Flandoli, B. Gess, and M. Scheutzow, *Synchronization by noise for order-preserving random dynamical systems*, Ann. Probab. **45** (2017), no. 2, 1325–1350.

7. I. I. Gikhman and A. V. Skorokhod, *The Theory of Stochastic Processes. II*, Nauka, Moscow, 1973. [Translated from the Russian by S. Kotz. Corrected printing of the first edition. Berlin: Springer, 2004. viii, 441 p.].

8. K. Itô and M. Nisio, *On stationary solutions of a stochastic differential equation*, J. Math. Kyoto Univ. **4** (1964), no. 1, 1–75.

9. R. Z. Khasminskii. *On positive solutions of the equation $\mathfrak{A}u + vu = 0$*, Theory of Probability and Its Applications **4** (1959), no. 3, 309–318.

10. R.Z. Khasminskii, *Stochastic Stability of Differential Equations*, Springer, Berlin, Heidelberg, 2012. [Originally published in Russian, by Nauka, Moskow 1969. 1st English ed. published 1980 under R.Z. Has'minski in the series Mechanics: Analysis by Sijthoff & Noordhoff.].

11. A. Kulik, *Ergodic Behavior of Markov Processes: With Applications to Limit Theorems*. De Gruyter Studies in Mathematics Series. Walter de Gruyter GmbH, 2017.

12. G. Da Prato and Z. Zabczyk, *Ergodicity for Infinite-Dimensional Systems*, Probability theory and mathematical statistics. Cambridge University Press, Cambridge, 1996.

13. P. E. Protter, *Stochastic Integration and Differential Equations*, Springer-Verlag, Berlin, 2004.

14. D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin, 1999.

15. A.-S. Sznitman, *Brownian Motion, Obstacles and Random Media*, Springer Monographs in Mathematics, Berlin: Springer, 1998.

16. A. Y. Veretennikov, *On strong solutions and explicit formulas for solutions of stochastic integral equations*, Math. USSR Sborn. **39** (1981), no. 3, 387–403.

INSTITUTE OF GEOPHYSICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 32 PALLADIN AVE., 03142, KYIV, UKRAINE; NATIONAL TECHNICAL UNIVERSITY OF UKRAINE "IGOR SIKORSKY KYIV POLITECHNIC INSTITUTE", KYIV, UKRAINE

E-mail address: `oaryasova@gmail.com`

INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 3 TERESHCHENKIVSKA STR., 01601, KYIV, UKRAINE; NATIONAL TECHNICAL UNIVERSITY OF UKRAINE "IGOR SIKORSKY KYIV POLITECHNIC INSTITUTE", KYIV, UKRAINE

E-mail address: `pilipenko.ay@gmail.com`