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REPRESENTATIONS OF THE FINITE-DIMENSIONAL POINT DENSITIES IN ARRATIA FLOWS WITH DRIFT

We derive representations for finite-dimensional densities of the point process associated with an Arratia flow with drift in terms of conditional expectations of the stochastic exponentials appearing in the analog of the Girsanov theorem for the Arratia flow.

1. INTRODUCTION

The study of the point process associated with an Arratia flow $\{X^a(u, t) | u \in [0; 1], t \in [0; T]\}$ with drift a [1, Section 7] is carried out in the present paper by means of special (n, k) -point densities $p_t^{a, n, k}$, $k \leq n$. Such densities constitute a generalization of those discussed in [9, 11, 12] and are informally defined via the formula

$$P\left(\forall i = \overline{1, n} \ X^a(u_i, t) \in \bigcup_{j=1}^k [y_j; y_j + dy_j], \ \forall j = \overline{1, k} \ \{X^a(u_l, t) \mid l = \overline{1, n}\} \cap [y_j; y_j + dy_j] \neq \emptyset\right) = p_t^{a, n, k}(u; y) dy_1 \dots dy_k,$$

the strict definition to be provided later in the text.

We find the Radon–Nikodym representation for $p_t^{a, n, k}$ in terms of $p_t^{0, n, k}$. It is known [1, p. 194] that the distribution of the random process $(X^a(u_1, \cdot), \dots, X^a(u_n, \cdot))$ is absolutely continuous in $(C([0; T]))^n$ w.r.t. the distribution of $(X^0(u_1, \cdot), \dots, X^0(u_n, \cdot))$ with density

$$(1) \quad \tilde{\mathcal{E}}_{T, n}^a(u) = \exp \left\{ \sum_{k=1}^n \int_0^{\tau_k} a(X^0(u_k, t)) dX^0(u_k, t) - \frac{1}{2} \sum_{k=1}^n \int_0^{\tau_k} a^2(X^0(u_k, t)) dt \right\},$$

where $\tau_1 = T$ and

$$\tau_k = \inf \left\{ t \mid \prod_{j=1}^{k-1} (X^0(u_j, t) - X^0(u_k, t)) = 0 \right\} \wedge T, \quad k = \overline{1, n},$$

where $\inf \emptyset = +\infty$ by definition. Moreover, the distribution of an Arratia flow with a bounded Lipschitz continuous drift a as a random element in $D([0; 1], C([0; T]))$ is absolutely continuous w.r.t. the distribution of the Arratia flow with zero drift [1, Theorem 7.3.1].

Since the definition of the densities $p_t^{a, n, k}$ contains the condition for the flow to hit the neighborhoods of certain points at time t , we firstly investigate the distribution of (1) conditional on $(X^0(u_1, T), \dots, X^0(u_n, T))$.

Hereinafter the superscript $a = 0$ is dropped in the case of zero drift, and a is always assumed to be bounded and Lipschitz continuous. We write $x = (x_1, \dots, x_n)$ for points in \mathbb{R}^n , $n \in \mathbb{N}$.

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2. ON BROWNIAN BRIDGES AND RELATED CONDITIONAL DISTRIBUTIONS

Put $\Delta_n = \{u \in \mathbb{R}^n \mid u_1 < \dots < u_n\}$, $n \in \mathbb{N}$. The following constructive scheme is used. Assume $W = (w_1, \dots, w_n)$ to be a standard Wiener process in \mathbb{R}^n started at 0. Put $\tilde{w}_1 = w_1, \theta_1(u) = T$ and define

$$\theta_k(u) = \inf \{t \mid \tilde{w}_{k-1}(t) + u_{k-1} = w_k(t) + u_k\} \wedge T,$$

$$\tilde{w}_k(t) = -u_k + (u_k + w_k(t))\mathbb{I}(t < \theta_k(u)) + (u_{k-1} + \tilde{w}_{k-1}(t))\mathbb{I}(t \geq \theta_k(u)), \quad k = \overline{2, n},$$

where $u \in \Delta_n$. Denote $\tilde{W} = (\tilde{w}_1, \dots, \tilde{w}_n)$. Then one can easily verify the following statement.

Lemma 2.1. *In $(C([0; T]))^n$*

$$(X(u_1, \cdot), \dots, X(u_n, \cdot)) \stackrel{d}{=} u + \tilde{W},$$

and the expression $\tilde{\mathcal{E}}_{T,n}^a(u)$ in (1) has the same distribution as

$$\mathcal{E}_{T,n}^a(W, u) = \exp \left\{ \sum_{k=1}^n \int_0^{\theta_k(u)} a(u_k + w_k(t)) dw_k(t) - \frac{1}{2} \sum_{k=1}^n \int_0^{\theta_k(u)} a^2(u_k + w_k(t)) dt \right\}.$$

The Itô stochastic integrals that participate in the definition of $\mathcal{E}_{T,n}^a(W, u)$ can be expressed in terms of stochastic integrals w.r.t. the Brownian bridges $\eta = (\eta_1, \dots, \eta_n)$ associated with W :

$$(2) \quad w_k(t) = \frac{t}{T} w_k(T) - \eta_k(t), \quad t \in [0; T], k = \overline{1, n}.$$

We refer to [6, §5.6.B][10, pp. 299-300] for the general exposition of the theory of Brownian bridges.

Define the filtration

$$\mathcal{F}_t = \sigma(\eta_k(s), s \leq t, k = \overline{1, n}), \quad t \in [0; T],$$

which is further supposed to be augmented in a standard way. Each η_k is the solution of the SDE

$$(3) \quad \begin{aligned} d\eta_k(t) &= d\beta_k(t) - \frac{\eta_k(t)}{T-t} dt, \quad t \in [0; T), \\ \eta_k(0) &= 0, \end{aligned}$$

where every β_k is a $(\mathcal{F}_t)_{t \in [0; T]}$ -Wiener process and $\eta_k(T) = 0$. At the same time, every η_k admits the representation

$$(4) \quad \eta_k(t) = (T-t)b_k \left(\frac{t}{T(T-t)} \right), \quad t \in [0; T),$$

with b_1, \dots, b_n being independent standard Wiener processes.

Since

$$(5) \quad \mathbb{E} \int_0^T \frac{|\eta_1(t)|}{T-t} dt \leq \int_0^T \frac{(\mathbb{E} \eta_1^2(t))^{1/2}}{T-t} dt \leq \int_0^T \left(\frac{t}{(T-t)T} \right)^{1/2} dt < +\infty,$$

each Brownian bridge η_k is a semimartingale w.r.t its own filtration and the stochastic integrals w.r.t. the coordinates of η are defined in a usual way, at least for bounded progressively measurable integrands (see [4] for the full characterization of possible integrands).

For any $u \in \Delta^n, y \in \mathbb{R}^n$ define the following random process $\eta^{u,y} = (\eta_1^{u,y}, \dots, \eta_n^{u,y})$ in \mathbb{R}^n :

$$\eta^{u,y}(t) = \eta(t) + \left(1 - \frac{t}{T}\right)u + \frac{t}{T}y, \quad t \in [0; T],$$

and put

$$\begin{aligned}\theta_{ij}(u) &= \inf \{t \mid w_i(t) + u_i = w_j(t) + u_j\} \wedge T, \\ \theta_{ij}(u, y) &= \inf \{t \mid \eta_i^{u,y}(t) = \eta_j^{u,y}(t)\} \wedge T, \\ j &= \overline{1, i-1}, i = \overline{2, n}, y \in \mathbb{R}^n.\end{aligned}$$

Additionally, put

$$\theta_{kk}(u) = \theta_{jj}(u, y) = T, \quad k, j = \overline{1, n}.$$

Then a.s.

$$\theta_{ij}(u) = \theta_{ij}(u, W(T) + u), \quad j = \overline{1, i}, i = \overline{1, n}.$$

Lemma 2.2. *For all $u \in \Delta_n$ with probability 1 all $\theta_{ij}(u)$ that are less than T are distinct. For all $u \in \Delta_n$ and $y \in \mathbb{R}^n$ with probability 1 all $\theta_{ij}(u, y)$ that are less than T are distinct.*

Proof. The first assertion is trivial. The second one can be deduced from (4). \square

To describe sequences of collisions in finite-dimensional motions of an Arratia flow, we use the following construction that was introduced in [3, Definition 1.2] and is a reformulation of the one presented in [2, pp. 433-434]. Put

$$\begin{aligned}Sh_{n,k} &= \{(j_1, \dots, j_k) \mid j_i \in \{1, \dots, n-i\}, i = \overline{1, k}\}, \quad k = \overline{1, n}, \\ Sh_n &= \{\emptyset\} \cup \bigcup_{k=\overline{1, n-1}} Sh_{n,k}, \quad n \in \mathbb{N}.\end{aligned}$$

Recall $\widetilde{W} + u = (\widetilde{w}_1 + u_1, \dots, \widetilde{w}_n + u_n)$ to be coalescing Wiener processes constructed from the process $W + u$. Let $n - \varkappa$ be the number of distinct values in the sequence $\{\widetilde{w}_i(T) + u_i \mid i = \overline{1, n}\}$, \varkappa ranging in $\{0, \dots, n-1\}$. Let $\tau_1 < \tau_2 < \dots < \tau_\varkappa$ be random moments such that

$$\{\tau_1, \dots, \tau_\varkappa\} = \{\theta_k(u) \mid \theta_k(u) < T, k = \overline{1, n}\}.$$

By virtue of Lemma 2.2, such $\tau_1, \dots, \tau_\varkappa$ exist a.s.. Put $j_1 = \min\{i \mid \exists j \neq i \ \widetilde{w}_j(\tau_1) + u_j = \widetilde{w}_i(\tau_1) + u_i\}$ and define the process \widetilde{W}^{n-1} by excluding the j_1 -th coordinate from the vector $\widetilde{w} + u$. Then put $j_2 = \min\{i \mid \exists j \neq i \ \widetilde{w}_j^{n-1}(\tau_2) = \widetilde{w}_i^{n-1}(\tau_2)\}$, define \widetilde{W}^{n-2} by excluding the j_2 -th coordinate from the process \widetilde{W}^{n-1} and repeat the procedure until a random collection $S(W + u) = (j_1, \dots, j_\varkappa) \in Sh_{n,\varkappa}$ appears. We will call $S(W + u)$ the coalescing scheme for the process $W + u$.

Moreover, there exist nonrandom numbers $\{\lambda_{ij}(s) \mid i = 1, 2, j = \overline{1, n}\}$ such that a.s.

$$\theta_k(u) \mathbb{I}(S(W + u) = s) = \theta_{\lambda_{1k}(s)\lambda_{2k}(s)}(u) \mathbb{I}(S(W + u) = s), \quad k = \overline{1, n}.$$

The collection $\{\lambda_{ij}(s) \mid i = 1, 2, j = \overline{1, n}\}$ is completely determined by the value of the coalescence scheme s and can be restored from the latter directly, though, due to cumbrousness of the corresponding relations, we omit giving an explicit representation.

For ease of the further presentation, put

$$\begin{aligned}a_k(t, u, y, s) &= \mathbb{I}(t \leq \theta_{\lambda_{1k}(s)\lambda_{2k}(s)}(u, y)) \cdot a(\eta_k^{u,y}(t)), \\ t &\in [0; T], \quad k = \overline{1, n}, \quad y \in \mathbb{R}^n, \quad s \in Sh_n; \\ \mathfrak{e}_{T,n}^a(u, y, s) &= \exp \left\{ \sum_{k=1}^n \int_0^T a_k(t, y, s) d\beta_k(t) + \right. \\ &\quad \left. + \sum_{k=1}^n \int_0^T a_k(t, u, y, s) \left(\frac{y_k - u_k}{T} - \frac{\eta_k(t)}{T-t} - \frac{1}{2} a_k(t, u, y, s) \right) ds \right\}, \\ y &\in \mathbb{R}^n, \quad s \in Sh_n.\end{aligned}$$

Lemma 2.3. $\forall C > 0 \forall k = \overline{1, n}$

$$\mathbb{E} e^{C \int_0^T \frac{|\eta_k(t)|}{T-t} dt} < +\infty.$$

Proof. By (5) the process $t \mapsto \frac{\eta_k(t)}{T-t}$ is a Gaussian random element in $L_1([0; T])$, therefore the claim follows from the Fernique theorem [8, Theorem 3.1]. \square

Lemma 2.4. $\forall u \in \Delta_n \forall y \in \mathbb{R}^n \forall s \in Sh_n \forall p \geq 0$

$$\mathbb{E} (\mathbf{e}_{T,n}^a(u, y, s))^p \leq C_1 e^{C_2 \|y\|},$$

where

$$C_1 = e^{np|2p-1|T \cdot \|a\|_{L_\infty(\mathbb{R})}^2 + pn^{1/2} \|u\| \cdot \|a\|_{L_\infty(\mathbb{R})}} \left(\mathbb{E} e^{2p\|a\|_{L_\infty(\mathbb{R})} \int_0^T \frac{|\eta_k(t)|}{T-t} dt} \right)^{n/2},$$

$$C_2 = pn^{1/2} \|a\|_{L_\infty(\mathbb{R})}.$$

Proof. By the Cauchy inequality,

$$\begin{aligned} & (\mathbb{E} (\mathbf{e}_{T,n}^a(u, y, s))^p)^2 \leq \\ & \leq \left[\mathbb{E} \exp \left\{ \sum_{k=1}^n \int_0^T 2pa_k(t, y, s) d\beta_k(t) - \frac{1}{2} \sum_{k=1}^n \int_0^T (2pa_k(t, y, s))^2 dt \right\} \right] \times \\ & \quad \times \left[\mathbb{E} \exp \left\{ \sum_{k=1}^n \int_0^T \left(p(2p-1)a_k^2(t, y, s) - \frac{2pa_k(t, y, s)\eta_k(t)}{T-t} + \right. \right. \right. \\ & \quad \left. \left. \left. + \frac{2p(y_k - u_k)}{T} a_k(t, y, s) \right) dt \right\} \right] \leq \\ & \leq \exp \left\{ 2C_2 \|y\| + 2np|2p-1|T \cdot \|a\|_{L_\infty(\mathbb{R})}^2 + 2pn^{1/2} \|u\| \cdot \|a\|_{L_\infty(\mathbb{R})} \right\} \times \\ & \quad \times \mathbb{E} \exp \left\{ 2p\|a\|_{L_\infty(\mathbb{R})} \sum_{k=1}^n \int_0^T \frac{|\eta_k(t)|}{T-t} dt \right\}, \end{aligned}$$

thus the application of Lemma 2.3 finishes the proof. \square

In parallel to $S(W + u)$ one defines $S(\eta^{u,y})$, $y \in \mathbb{R}^n$, by applying the same recursive procedure to the process $\eta^{u,y}$ and the times $\theta_{ij}(u, y)$, $j = \overline{1, i-1}$, $i = \overline{2, n}$.

Theorem 2.1. $\forall u \in \Delta_n \forall y \in \mathbb{R}^n \forall s \in Sh_n$

$$\mathbb{E} (\mathbb{I}(S(W + u) = s) \mathcal{E}_{T,n}^a(W, u) / W(T) = y - u) = \mathbb{E} \mathbb{I}(S(\eta^{u,y}) = s) \mathbf{e}_{T,n}^a(u, y, s).$$

Proof. Suppose u is fixed. Define, for $k = \overline{1, n}$,

$$a_k^m(t, y, s) = \sum_{j=0}^{m-1} \mathbb{I}(t \in (\frac{j}{m}T; \frac{j+1}{m}T]) a_k(\frac{j}{m}T, u, y, s), \quad t \in [0; T], m \in \mathbb{N}.$$

Note that on the set $\{S(W + u) = s\}$ for $s \in Sh_n$

$$a(w_k(t) + u_k) \mathbb{I}(t \leq \theta_k(u)) = a_k(t, u, W(T) + u, s), \quad k = \overline{1, n}, t \in [0; T].$$

Due to the representation (2)

$$\begin{aligned} (6) \quad & \int_0^T a_k^m(t, w(T) + u, s) dw_k(t) = \frac{w_k(T)}{T} \int_0^T a_k^m(t, w(T) + u, s) dt + \\ & + \sum_{j=1}^{m-1} a_k(\frac{j}{m}T, u, y, s) (\eta_k(\frac{j+1}{m}T) - \eta_k(\frac{j}{m}T)). \end{aligned}$$

Since $W(T)$ and η are independent, the application of the disintegration theorem gives, due to (6), that

$$(7) \quad \mathbb{E}(\mathbb{I}(S(W+u) = s)\alpha_m(u)/W(T) = y-u) = \mathbb{E}\mathbb{I}(S(\eta^{u,y}) = s)e_m(u, y, s),$$

where

$$\begin{aligned} \alpha_m(u) &= \exp \left\{ \sum_{k=1}^n \int_0^T a_k^m(t, W(T) + u, s) dw_k(t) - \right. \\ &\quad \left. - \frac{1}{2} \sum_{k=1}^n \int_0^T (a_k^m(t, W(T) + u, s))^2 dt \right\}, \\ e_m(u, y, s) &= \exp \left\{ \sum_{k=1}^n \sum_{j=1}^{m-1} a_k \left(\frac{j}{m} T, u, y, s \right) (\eta_k \left(\frac{j+1}{m} T \right) - \eta_k \left(\frac{j}{m} T \right)) + \right. \\ &\quad \left. + \sum_{k=1}^n \int_0^T a_k^m(t, y, s) \left(\frac{y_k - u_k}{T} - \frac{1}{2} a_k^m(t, y, s) \right) dt \right\}. \end{aligned}$$

Since the functions $a_k, k = \overline{1, n}$, are piecewise continuous a.s., there exists a set Ω' of full probability such that $\forall \omega \in \Omega'$ for almost all t in $[0; T]$

$$(8) \quad a_k^m(t, W(T) + u, s) \xrightarrow[m \rightarrow \infty]{} a_k(t, u, W(T) + u, s) \text{ in } \mathbb{R}.$$

It holds due to (8) and the boundedness of the function a that, for $k = \overline{1, n}$,

$$\begin{aligned} \mathbb{E} \int_0^T &\left((a_k^m(t, W(T) + u, s) - a_k(t, u, W(T) + u, s))^2 + \right. \\ &\quad \left. + |a_k^m(t, W(T) + u, s) - a_k(t, u, W(T) + u, s)| \frac{|\eta_k(t)|}{T-t} \right) dt \xrightarrow[m \rightarrow \infty]{} 0, \end{aligned}$$

hence

$$(9) \quad \log e_m(u, y, s) \Big|_{y=W(T)+u} \xrightarrow[m \rightarrow \infty]{\text{P}} \log \mathcal{E}_{T,n}^a(u, y, s) \Big|_{y=W(T)+u}.$$

and

$$(10) \quad \log \alpha_m(u) \xrightarrow[m \rightarrow \infty]{\text{P}} \log \mathcal{E}_{T,n}^a(W, u).$$

Repeating the reasoning of Lemma 2.4 one checks that given y, s the estimate of Lemma 2.4 holds for each $e_m(u, y, s)$, so the sequence

$$\left\{ \mathbb{I}(S(\eta^{u,y}) = s) e_m(u, y, s) \Big|_{y=W(T)+u} \right\}_{m \in \mathbb{N}}$$

is uniformly integrable. The boundedness of the function a implies the uniform integrability of the sequence $\{\mathbb{I}(S(W+u) = s)\alpha_m(u)\}_{m \in \mathbb{N}}$. Thus the claim of the theorem follows from (9), (10) and (7). \square

Lemma 2.5. $\forall u \in \Delta_n \forall y \in \mathbb{R}^n \forall s \in Sh_n$

$$\mathbb{I}(S(\eta^{u,y_m}) = s) \xrightarrow[m \rightarrow \infty]{a.s.} \mathbb{I}(S(\eta^{u,y}) = s),$$

whenever $y_m \rightarrow y, m \rightarrow \infty$.

Proof. We assume that u and y are fixed throughout the proof. Due to Lemma 2.2 it is sufficient to check that a particular ordering of the moments $\theta_{ij}(u, y)$ is preserved in a sufficiently small random neighborhood of the point y .

The representation (4) implies that, with $f(s) = \frac{T^2 s}{T s + 1}$,

$$\begin{aligned}\theta_{ij}(u, y) &= f(s_{ij}(y)), \\ s_{ij}(y) &= \inf \left\{ s \geq 0 \mid b_i(s) - b_j(s) + s(y_i - y_j) + \frac{u_i - u_j}{T} = 0 \right\}, \\ j &= \overline{1, i-1}, i = \overline{2, n}, y \in \mathbb{R}^n,\end{aligned}$$

where the convention $f(\infty) = T$ is adopted. Since $\eta^{u,y}(T) = y$, the probability

$$P \left(\exists k \neq j : \eta_k^{u,y}(t) \neq \eta_j^{u,y}(t), t \in [0; T], \eta_k^{u,y}(T) = \eta_j^{u,y}(T) \right)$$

can be greater than 0 only if $y_j = y_k$ for some $k, j, k \neq j$. However, in that case

$$\begin{aligned}P \left(\eta_k^{u,y}(t) \neq \eta_j^{u,y}(t), t \in [0; T], \eta_k^{u,y}(T) = \eta_j^{u,y}(T) \right) &= \\ &= P \left(\forall s \geq 0 \ b_k(s) - b_j(s) + \frac{u_k - u_j}{T} \neq 0 \right) = 0.\end{aligned}$$

Therefore with probability 1 for each (k, j) either $\theta_{kj}(u, y) < T$ or

$$(11) \quad \inf_{t \in [0; T]} |\eta_k^{u,y}(t) - \eta_j^{u,y}(t)| > 0.$$

From now on, only a set of full probability which the condition (11) or its counterpart holds for is considered.

Fix a pair (k, j) and some positive $\varepsilon \ll 1$. Suppose $\theta_{kj}(u, y) < T$. Obviously, there exists random $r > 0$ such that the condition $\|y - y'\| < r$ implies $\theta_{kj}(u, y') \geq \theta_{kj}(u, y) - \varepsilon$. The moment $\theta_{kj}(u, y)$ is a Markov time w.r.t. the filtration generated by the process η , therefore it follows from the iterated logarithm law for the Wiener process that there exist random $\varepsilon_1, \varepsilon_2 : 0 < \varepsilon_1, \varepsilon_2 < \varepsilon$ such that

$$\begin{aligned}(12) \quad \eta_k(\theta_{kj}(u, y) + \varepsilon_1) - \eta_j(\theta_{kj}(u, y) + \varepsilon_1) + (u_k - u_j) \left(1 - \frac{\theta_{kj}(u, y)}{T} \right) + (y_k - y_j) \frac{\theta_{kj}(u, y)}{T} + \\ + \varepsilon_1 \left(\frac{y_k - y_j}{T} + \frac{u_j - u_k}{T} \right) > 0\end{aligned}$$

and

$$\begin{aligned}(13) \quad \eta_k(\theta_{kj}(u, y) + \varepsilon_2) - \eta_j(\theta_{kj}(u, y) + \varepsilon_2) + (u_k - u_j) \left(1 - \frac{\theta_{kj}(u, y)}{T} \right) + (y_k - y_j) \frac{\theta_{kj}(u, y)}{T} + \\ + \varepsilon_2 \left(\frac{y_k - y_j}{T} + \frac{u_j - u_k}{T} \right) < 0.\end{aligned}$$

Thus in order to have $\theta_{kj}(u, y') \leq \theta_{kj}(u, y) + \varepsilon$ when $\|y - y'\| \leq \delta$ it is sufficient to choose δ in such a way that the signs in (12) and (13) do not change when y_k and y_j are replaced with y'_k and y'_j , respectively. \square

By using Lemmas 2.4 and 2.5 one can establish the following result.

Lemma 2.6. $\forall u \in \Delta_n \ \forall s \in Sh_n$ the function $\mathbb{R}^n \ni y \mapsto \mathbb{E} \mathbb{I}(S(\eta^{u,y}) = s) \mathbf{e}_{T,n}^a(u, y, s)$ is continuous.

3. FINITE-DIMENSIONAL DENSITIES FOR THE POINT PROCESS IN THE ARRATIA FLOW

The section is devoted to the study of the connection between the finite-dimensional densities for the Arratia flow and the stochastic exponentials that arise when addressing the Girsanov theorem for Arratia flows with drift [1, Section 7]. We start with the

corresponding results from [1, §§7.2-7.3]. Consider a dense subset of $[0; 1]$, $U = \{u_k \mid k \in \mathbb{N}\}$. Given an Arratia flow X define

$$\begin{aligned}\tau_1 &= T, \\ \tau_k &= \inf \left\{ s \mid \prod_{j=1}^{k-1} (X(u_k, s) - X(u_j, s)) = 0 \right\} \wedge T, \quad k \geq 2,\end{aligned}$$

and put, for $u^{(n)} = (u_1, \dots, u_n)$,

$$\begin{aligned}I_n(u^{(n)}) &= \sum_{k=1}^n \int_0^{\tau_k} a(X(u_k, t)) dX(u_k, t), \\ J_n(u^{(n)}) &= \sum_{k=1}^n \int_0^{\tau_k} a^2(X(u_k, t)) dt, \quad n \in \mathbb{N}.\end{aligned}$$

The integrals in the expression for I_n are ordinary Itô integrals w.r.t. the Wiener processes $X(u_j, \cdot)$, $j \in \mathbb{N}$. Note I_n and J_n are well defined as functions on \mathbb{R}^n . There exist limits

$$\begin{aligned}I &= L_2\text{-} \lim_{n \rightarrow \infty} I_n(u^{(u)}), \\ J &= L_2\text{-} \lim_{n \rightarrow \infty} J_n(u^{(u)}),\end{aligned}$$

which do not depend on the set U . The distribution of an Arratia flow with drift a as a random element in the Skorokhod space $D([0; 1], C([0; T]))$ is absolutely continuous w.r.t. the distribution of X with density

$$\tilde{\mathcal{E}}_T^a = \exp \left\{ I - \frac{1}{2} J \right\}.$$

Recall that $\tilde{\mathcal{E}}_{T,n}^a(u^{(n)}) = \exp\{I_n(u^{(n)}) - J_n(u^{(n)})\}$, $n \in \mathbb{N}$.

Lemma 3.1. $\forall n \in \mathbb{N}$

$$\begin{aligned}\mathbb{E} \left(\tilde{\mathcal{E}}_{T,m}^a(u^{(m)}) / X(u_1, \cdot), \dots, X(u_n, \cdot) \right) &= \tilde{\mathcal{E}}_{T,n}^a(u^{(n)}), \quad m \geq n, \\ \mathbb{E} \left(\tilde{\mathcal{E}}_T^a / X(u_1, \cdot), \dots, X(u_n, \cdot) \right) &= \tilde{\mathcal{E}}_{T,n}^a(u^{(n)}).\end{aligned}$$

Proof. The random variables $\{\tilde{\mathcal{E}}_{T,n}^a(u^{(n)})\}_{n \in \mathbb{N}}$ form a uniformly integrable sequence (see [1, the proof of Theorem 7.3.1, pp. 268-270]), therefore it is sufficient to prove

$$\mathbb{E} \left(\tilde{\mathcal{E}}_{T,m}^a(u^{(m)}) / \mathcal{G}_n \right) = \tilde{\mathcal{E}}_{T,n}^a(u^{(n)}), \quad m \geq n,$$

where $\mathcal{G}_n = \sigma(X(u_1, \cdot), \dots, X(u_n, \cdot))$. Suppose n and $m > n$ are fixed. Put

$$e_k(t) = \exp \left\{ \int_0^t a(X(u_k, s)) dX(u_k, s) - \frac{1}{2} \int_0^t a^2(X(u_k, s)) ds \right\}, \quad t \in [0; T], k \in \mathbb{N},$$

then, using the Itô formula one can verify that, for $k \in \mathbb{N}$,

$$e_k(\tau_k) = 1 + \int_0^{\tau_k} e_k(t) a(X(u_k, t)) dX(u_k, t),$$

since every τ_k is a stopping time w.r.t. the filtration generated by the processes $X(u_j, \cdot)$, $j \in \mathbb{N}$. Therefore

$$\begin{aligned} \mathbb{E} \left(\tilde{\mathcal{E}}_{T,m}^a \left(u^{(m)} \right) / \mathcal{G}_n \right) &= \tilde{\mathcal{E}}_{T,n}^a \left(u^{(n)} \right) \mathbb{E} \left(\prod_{j=n+1}^m e_j / \mathcal{G}_n \right) = \\ &= \tilde{\mathcal{E}}_{T,n}^a \left(u^{(n)} \right) \left(1 + \mathbb{E} \left(\sum_{k=1}^{m-n} \sum_{j_1 < \dots < j_k} A_{j_1 \dots j_k} / \mathcal{G}_n \right) \right), \end{aligned}$$

where

$$A_{j_1 \dots j_k} = \prod_{l=1}^k \int_0^{\tau_{j_l}} a(X(u_{j_l}, t)) e_{j_l}(t) dX(u_{j_l}, t).$$

Rewriting the multipliers in every $A_{j_1 \dots j_k}$ as

$$\int_0^T a_{j_l}(t) dX(u_{j_l}, t), \quad l = \overline{1, k},$$

for some progressively measurable w.r.t. the filtration generated by $X(u_j, \cdot)$, $j \in \mathbb{N}$, processes

$$a_{j_l}(t) = \mathbb{I}(t \leq \tau_{j_l}) a(X(u_{j_l}, t)) e_{j_l}(t),$$

applying the same approximation scheme as the one used in the proof of Theorem 2.1 and utilizing the fact that the joint covariance of $X(u_i, \cdot)$ and $X(u_j, \cdot)$ equals

$$(t - \inf \{s \mid X(u_i, s) = X(u_j, s)\})_+, \quad t \geq 0, i, j \in \mathbb{N},$$

one proves via standard reasoning that

$$\mathbb{E}(A_{j_1 \dots j_k} / \mathcal{G}_n) = 0,$$

which concludes the proof. \square

The following notation will be used further. Let $\{X^a(u, t) \mid u \in [0; 1], t \in [0; T]\}$ be an Arratia flow with drift a . Define

$$\begin{aligned} \mathfrak{X}_t^a(u) &= \{X^a(u_k, t) \mid k = \overline{1, n}\}, \\ \vec{X}^a(u, \cdot) &\equiv \vec{X}^a(u) = (X^a(u_1, \cdot), \dots, X^a(u_n, \cdot)), \quad u = (u_1, \dots, u_n) \in \mathbb{R}^n, n \in \mathbb{N}. \end{aligned}$$

Analogously to the case of coalescing Wiener processes in Section 2 one defines the coalescence scheme $S(\vec{X}^a(u))$ for the family $(X^a(u_1, \cdot), \dots, X^a(u_n, \cdot))$.

Given a set $K = \{k_1, \dots, k_m\} \subset \{1, \dots, n\}$ and a point $z \in \mathbb{R}^n$ we denote by z^{-K} the vector obtained by removing in the vector z all the coordinates whose numbers are in K ; by z^K , the vector obtained by removing all coordinates except those in K . We write $z^{K_1, \pm K_2}$ for $(z^{K_1})^{\pm K_2}$.

The following definitions of the finite-dimensional densities were introduced in [3] and represent a further development of the notions used in [9, 11].

Given an Arratia flow X^a , a starting point $u \in \Delta_n$ and a coalescence scheme $s \in Sh_{n,k}$ for some k the corresponding $(n-j)$ -point density $p_T^{a,n,s,n-j}(u; \cdot)$, $j \geq k$, is a measurable function on \mathbb{R}^{n-j} such that for any bounded nonnegative measurable $f : \mathbb{R}^{n-j} \rightarrow \mathbb{R}$

$$\mathbb{E} \sum_{\substack{v_1, \dots, v_{n-j} \in \mathfrak{X}_t^a(u), \\ v_1, \dots, v_{n-j} \text{ are distinct}}} f(v_1, \dots, v_{n-j}) \mathbb{I}(S(\vec{X}^a(u)) = s) = \int_{\mathbb{R}^{n-j}} p_t^{a,n,s,n-j}(u; y) f(y) dy.$$

The existence of $p_t^{0,n,s,n-j}(u; \cdot)$ is shown in [3, Lemma 3.1] whereas an explicit expression in the particular case $k = j$ is obtained in [3, Theorem 3.1].

Lemma 3.2. *For all $s \in Sh_{n,k}$, $u \in \Delta_n$ and $j \leq n - k$ the density $p_t^{a,n,s,j}(u; \cdot)$ exists.*

Proof. Let A be a Borel subset of Δ_j . By the Girsanov theorem for the Arratia flow

$$\begin{aligned} \mathbb{E} & \sum_{\substack{v_1, \dots, v_j \in \mathfrak{X}_t^a(u), \\ v_1, \dots, v_j \text{ are distinct}}} \mathbb{I}_A(v_1, \dots, v_j) \mathbb{I}(S(\vec{X}^a(u)) = s) = \\ & = \mathbb{E} \sum_{\substack{v_1, \dots, v_j \in \mathfrak{X}_t(u), \\ v_1, \dots, v_j \text{ are distinct}}} \mathbb{I}_A(v_1, \dots, v_j) \mathbb{I}(S(\vec{X}(u)) = s) \tilde{\mathcal{E}}_{T,n}^a(u) \leq \\ & \leq \sum_{\substack{L = \{l_1, \dots, l_j\}, \\ l_i \in \{1, \dots, n-k\}, \ i = \overline{1, j}}} \mathbb{E} \mathbb{I}_A(\vec{X}(u^L, t)) \mathbb{E}(\tilde{\mathcal{E}}_{T,n}^a(u) / \vec{X}(u^L, t)) \leq \\ & \leq \sum_{\substack{L = \{l_1, \dots, l_j\}, \\ l_i \in \{1, \dots, n-k\}, \ i = \overline{1, j}}} \int_A p_t^{0,j,j}(u^L; y) \mathbb{E}(\tilde{\mathcal{E}}_{T,n}^a(u) / \vec{X}(u^L, t) = y) dy. \end{aligned}$$

The case when A is not a subset of Δ_j is treated similarly. \square

Fix some $u \in \mathbb{R}^n$ and $k \in \{0, \dots, n-1\}$. Following [2, pp. 433-434], we associate with a coalescence scheme $s = (j_1, \dots, j_k)$ a partition of the set $\{1, \dots, n\}$ as follows. Starting from the partition consisting of singletons, at each step $i = 1, \dots, k$ proceed by merging two subsequent blocks in the current partition with the numbers j_i and $j_i + 1$, the blocks being listed in ascending order of their minimal elements. Let the blocks of the final partition be π_1, \dots, π_k . We define the set $I(s) = \{\min \pi_i \mid i = \overline{1, n-k}\}$. As a result,

$$|\{X^a(u_i, T) \mid i \in I(s)\}| = n - k$$

on $\{S(\vec{X}^a(u)) = s\}$. Obviously, the coalescence scheme deterministically and uniquely defines $I(s)$, which does not depend on u and a specific realization of the flow X^a .

Denote by $g_T^m(u; \cdot)$ the m -dimensional Gaussian density with mean u and variance $T \text{Id}_{m \times m}$, where $\text{Id}_{m \times m}$ is the unit square matrix of size m , $m \in \mathbb{N}$.

Theorem 3.1 (cf. [7]). *Assume $u \in \Delta_n$ and $s \in Sh_{n,n-k}$ for some $k \in \{0, \dots, n-1\}$. Then for each $j \in \{1, \dots, k\}$ for all $y \in \Delta_k$*

$$\begin{aligned} p_T^{a,n,s,j}(u; y) &= \sum_{\substack{L = \{l_1, \dots, l_j\} \subset \\ \{1, \dots, k\}}} g_T^j(u^{I(s),L}; z^{I(s),L}) \int_{\mathbb{R}^{k-j}} dz^{I(s),-L} g_T^{k-j}(u^{I(s),-L}; z^{I(s),-L}) \\ &\quad \int_{\mathbb{R}^{n-k}} dz^{-I(s)} g_T^{n-k}(u^{-I(s)}; z^{-I(s)}) (\mathbb{E} \mathbb{I}(S(\eta^{u,z}) = s) \mathfrak{e}_{T,n}^a(u, z, s)) \Big|_{\substack{z \in \mathbb{R}^n, \\ z^{I(s),L} = y}}. \end{aligned}$$

Proof. Fix u and take W and \widetilde{W} as in Section 2. Define

$$B_\delta^+(v) = [v; v + \delta], \quad v \in \mathbb{R}, \quad \delta > 0,$$

$$\xi = (\widetilde{W}(T) + u)^{I(s)},$$

$$\mathcal{W} = \{\widetilde{w}_1(T) + u_1, \dots, \widetilde{w}_n(T) + u_n\}.$$

We have, due to the Lebesgue differentiation theorem and [1, Theorem 7.3.1], for almost all y

$$\begin{aligned} p_T^{a,n,s,j}(u; y) &= \lim_{\delta \rightarrow 0+} \delta^{-j} \mathbb{E} \prod_{i=1}^j |B_\delta^+(y_i) \cap \mathfrak{X}_T^a(u)| \cdot \mathbb{I}(S(\vec{X}^a(u)) = s) = \\ &= \lim_{\delta \rightarrow 0+} \delta^{-j} \mathbb{E} \prod_{i=1}^j |B_\delta^+(y_i) \cap \mathcal{W}| \cdot \mathcal{E}_{T,n}^a(W, u) \mathbb{I}(S(W + u) = s), \end{aligned}$$

since by Lemma 2.1 $\vec{X}(u) \stackrel{d}{=} \widetilde{W} + u$. The reasoning of [9, Appendix B], combined with Lemma 2.4, allows one to replace $|B_\delta^+(y_i) \cap \mathcal{W}|$ with $\mathbb{I}(|B_\delta^+(y_i) \cap \mathcal{W}| = 1)$ for all i in the previous formula, so that

$$\begin{aligned} p_T^{a,n,s,j}(u; y) &= \\ &= \lim_{\delta \rightarrow 0+} \delta^{-j} \mathbb{E} \sum_{L=\{l_1, \dots, l_j\} \subset \{1, \dots, k\}} \prod_{i=1}^j \mathbb{I}_{B_\delta^+(y_i)}(\xi_{l_i}) \mathcal{E}_{T,n}^a(W, u) \mathbb{I}(S(W+u) = s). \end{aligned}$$

Consider a separate term

$$A_{\delta,L} = \mathbb{E} \prod_{i=1}^j \mathbb{I}_{B_\delta^+(y_i)}(\xi_{l_i}) \mathcal{E}_{T,n}^a(W, u) \mathbb{I}(S(W+u) = s).$$

Since on the set $\{S(W+u) = s\}$ by the definition of $I(s)$

$$\mathcal{W} = \{w_i(T) + u_i \mid i \in I(s)\},$$

Theorem 2.1 implies that

$$\begin{aligned} A_{\delta,L} &= \mathbb{E} \mathbb{E} \left(\prod_{i=1}^j \mathbb{I}_{B_\delta^+(y_i)}(\xi_{l_i}) \mathbb{I}(S(W+u) = s) \mathcal{E}_{T,n}^a(W, u) / W(T) \right) = \\ &= \mathbb{E} \prod_{i=1}^j \mathbb{I}_{B_\delta^+(y_i)} \left((W(T) + u)_i^{I(s),L} \right) \left(\mathbb{E} \mathbb{I}(S(\eta^{u,z}) = s) \mathcal{E}_{T,n}^a(u, z, s) \right) \Big|_{z=W(T)+u}. \end{aligned}$$

The vectors $W(T)^{I(s),L}, W(T)^{I(s),-L}$ and $W(T)^{-I(s)}$ being independent given fixed s , the local property of conditional expectation [5, Lemma 6.2] implies that

$$\begin{aligned} p_T^{a,n,s,j}(u; y) &= \lim_{\delta \rightarrow 0+} \delta^{-j} \mathbb{E} \sum_{L=\{l_1, \dots, l_j\} \subset \{1, \dots, k\}} \prod_{i=1}^j \mathbb{I}_{B_\delta^+(y_i)} \left((W(T) + u)_i^{I(s),L} \right) \times \\ &\quad \times \left(\mathbb{E} \mathbb{I}(S(\eta^{u,z}) = s) \mathcal{E}_{T,n}^a(u, z, s) \right) \Big|_{\substack{z^{I(s),L} = (W(T) + u)^{I(s),L}, \\ z^{I(s),-L} = \alpha, \\ z^{-I(s)} = \beta}}, \end{aligned}$$

where $W(T)^{I(s),L}$ and the Gaussian random variables $\alpha \sim \mathcal{N}(u^{I(s),-L}, T \cdot \text{Id}_{(k-j) \times (k-j)})$, $\beta \sim \mathcal{N}(u^{-I(s)}, T \text{Id}_{(n-k) \times (n-k)})$ are jointly independent. With Lemma 2.6, the rest of the proof follows by standard reasoning. We omit the details. \square

Given an Arratia flow X^a , a starting point $u \in \Delta_n$ and $k \in \{1, \dots, n\}$ the corresponding (n, k) -point density is a measurable function $p_T^{a,n,k}(u; \cdot)$ on \mathbb{R}^k such that for any bounded nonnegative measurable $f : \mathbb{R}^k \rightarrow \mathbb{R}$

$$(14) \quad \mathbb{E} \sum_{\substack{v_1, \dots, v_k \in \mathfrak{X}_t^a(u), \\ v_1, \dots, v_k \text{ are distinct}}} f(v_1, \dots, v_k) \mathbb{I}(|\mathfrak{X}_t^a(u)| \geq k) = \int_{\mathbb{R}^k} p_T^{a,n,k}(u; y) f(y) dy.$$

The next consequence of the formula of total probability gives a relation between $p_T^{a,n,k}$ and $p_T^{a,n,s,k}$.

Lemma 3.3. *For any $n \in \mathbb{N}, u \in \Delta_n$ and $k \in \{1, \dots, n\}$ a.e.*

$$p_T^{a,n,k}(u; \cdot) = \sum_{l=0}^{n-k} \sum_{s \in Sh_{n,l}} p_T^{a,n,s,k}(u; \cdot).$$

The k -point density $p_T^{a,k}(\cdot)$ (cf. [9, 11]) is defined as a measurable function on \mathbb{R}^k such that the analog of (14) holds with $\mathfrak{X}_T^a(u)$ replaced with the set $\{X^a(v, T) \mid v \in [0; 1]\}$ and the condition $|\mathfrak{X}_t^a(u)| \geq k$ dropped. The result of [3, Theorem 3.2] admits the following extension, the proof being the same with some minor changes having been made.

Theorem 3.2. *Let $u^{(n)} = (u_1^{(n)}, \dots, u_n^{(n)}) \in \Delta_n$, $n \in \mathbb{N}$, be such that $u_1^{(n)} = 0, u_n^{(n)} = 1, n \in \mathbb{N}$,*

$$\{u_1^{(n)}, \dots, u_n^{(n)}\} \subset \{u_1^{(n+1)}, \dots, u_{n+1}^{(n+1)}\}, \quad n \in \mathbb{N},$$

and

$$\max_{j=0, n-1} (u_{j+1}^{(n)} - u_j^{(n)}) \xrightarrow{n \rightarrow \infty} 0.$$

Then for all $k \in \mathbb{N}$ a.e.

$$p_T^{a,n,k}(u^{(n)}; \cdot) \nearrow p_T^{a,k}, \quad n \rightarrow \infty.$$

Due to Lemma 3.3, Theorem 3.1 provides an explicit expression for the densities $p_T^{a,k}(u; \cdot)$ in terms of conditional expectations of certain stochastic exponentials discussed in Section 2.

Consider $u \in \Delta_n$. Suppose that elements of the set $\mathfrak{X}_T(u)$ are listed in ascending order. Let \varkappa be the cemetery state. Given a set $L = \{l_1, \dots, l_k\}$, $l_i \in \mathbb{N}$, $i = \overline{1, k}$, for some k , put the random vector $\mathcal{X}_T^L(u)$ to be equal to

$$(15) \quad ((\mathfrak{X}_T(u))_{l_1}, \dots, (\mathfrak{X}_T(u))_{l_k}),$$

if $\max_{i=\overline{1, k}} l_i \leq |\mathfrak{X}_T(u)|$, and \varkappa , otherwise. We denote the density of $\mathcal{X}_T^L(u)$ in \mathbb{R}^k by $q_T^L(u; \cdot)$. Note that always

$$\int_{\mathbb{R}^k} q_T^L(u; y) dy < 1.$$

Theorem 3.3 (cf. [7]). *For any $u \in \Delta_n$ and any $k \in \{1, \dots, n\}$ a.e.*

$$p_T^{a,n,k}(u; y) = \sum_{L=\{l_1, \dots, l_k\}, l_i \in \mathbb{N}, i=\overline{1, k}} q_T^L(u; y) \mathbb{E} \left(\tilde{\mathcal{E}}_{T,n}^a(u) / \mathcal{X}_T^L(u) = y \right).$$

Proof. Proceeding similarly to the proof of Theorem 3.1, one obtains a.e.

$$\begin{aligned} p_T^{a,n,k}(u; y) &= \lim_{\delta \rightarrow 0+} \delta^{-k} \sum_{\substack{L=\{l_1, \dots, l_k\}, \\ l_i \in \mathbb{N}, i=\overline{1, k}}} \mathbb{E} \tilde{\mathcal{E}}_T^a(u) \mathbb{I} (\mathcal{X}_T^L(u) \neq \varkappa) \prod_{i=1}^k \mathbb{I}_{B_\delta^+(y_i)} ((\mathcal{X}_T^L(u))_i) = \\ &= \lim_{\delta \rightarrow 0+} \delta^{-k} \sum_{\substack{L=\{l_1, \dots, l_k\}, \\ l_i \in \mathbb{N}, i=\overline{1, k}}} \mathbb{E} \mathbb{I} (\mathcal{X}_T^L(u) \neq \varkappa) \times \mathbb{I} \left(\mathcal{X}_T^L(u) \in \bigcup_{i=1}^k B_\delta^+(y_i) \right) \times \\ &\quad \times \mathbb{E} \left(\tilde{\mathcal{E}}_T^a(u) / \mathcal{X}_T^L(u) \right). \end{aligned}$$

The application of the Lebesgue differentiation theorem finishes the proof. \square

Replacing in (15) the set $\mathfrak{X}_T(u)$ with the set $\{X(v, T) \mid v \in [0; 1]\}$ one defines, analogously to $\mathcal{X}_T^L(u)$, the random vector \mathcal{X}_T^L with values in $\mathbb{R}^k \cup \{\varkappa\}$. The corresponding density being denoted by $q_T^L(\cdot)$, the following result holds.

Theorem 3.4. *For any $k \in \mathbb{N}$ a.e.*

$$p_T^{a,k}(y) = \sum_{L=\{l_1, \dots, l_k\}, l_i \in \mathbb{N}, i=\overline{1, k}} q_T^L(y) \mathbb{E} \left(\tilde{\mathcal{E}}_T^a / \mathcal{X}_T^L = y \right).$$

The arguments are repetitive and thus omitted.

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REFERENCES

1. A.A.Dorogovtsev, *Meroznachnye protsessy i stokhasticheskie potoki*, (in Russian) [Measure-valued Processes and Stochastic Flows], Proceedings of Institute of Mathematics of NAS of Ukraine, Mathematics and its Applications **66**, Kiev, 2007.
2. A.A.Dorogovtsev, *Krylov-Veretennikov expansion for coalescing stochastic flows*, Commun. Stoch. Anal. **6** (2012), no. 3, 421–435.
3. A.A.Dorogovtsev, M.B.Vovchanskii, *On approximations of the point measures associated with the Brownian web by means of the fractional step method and the discretization of the initial interval*, Ukr. Math. J. **72** (2020), no 9, 1179–1194.
4. P.Gosselin, T.Wurzbacher, *An Itô type isometry for loops in \mathbf{R}^d via the Brownian bridge*, Séminaire de probabilités de Strasbourg **31** (1997), 225–231.
5. O.Kallenberg, *Foundations of Modern Probability*, 2nd ed., Probability and its Applications, Springer-Verlag, New York, 2002.
6. I.Karatzas, S.E.Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed., Springer-Verlag, 1996.
7. V.Konarovskyi, V.Marx, *Conditional Distribution of Independent Brownian Motions to Event of Coalescing Paths*, preprint, available at <https://arxiv.org/abs/2008.02568> (2020).
8. H.-H.Kuo, *Gaussian measures in Banach spaces*, Lecture Notes in Mathematics **463**, Springer-Verlag, Berlin, 1975.
9. R.Munasinghe, R.Rajesh, R.Tribe, O.Zaboronski, *Multi-scaling of the n -point density function for coalescing Brownian motions*, Communications in Mathematical Physics **268** (2006), 717–725.
10. Ph.E.Protter, *Stochastic integration and differential equations*, 2nd ed, Applications of Mathematics (New York) **21**, Stochastic Modelling and Applied Probability, Springer-Verlag, Berlin, 2004. xiv+415 pp.
11. R.Tribe, O.Zaboronski, *Pfaffian formulae for one dimensional coalescing and annihilating systems*, Electronic Journal of Probability **16** (2011), 2080–2103.
12. V.Fomichov, *The distribution of the number of clusters in the Arratia flow*, Communications on Stochastic Analysis **10** (2016), no. 3, 257–270.

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