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ON THE PROPERTIES OF MULTIVARIATE ISOTROPIC RANDOM FIELDS ON THE BALL

We consider multivariate isotropic random fields on the ball $\mathbb{B}^d$. We first study their regularity properties in terms of Sobolev spaces. We further derive conditions guaranteeing the Hölder continuity of their covariance kernels and we prove the existence of sample Hölder continuous modifications for Gaussian random fields. Furthermore, we measure the error of truncated approximations of the corresponding series' representations. Moreover our developments are supported by numerical experiments. The majority of our results are new for multivariate random fields indexed over other domains, too. We express some of them for the case of the sphere.

1. Introduction

A plenty of phenomena in Astrophysics, Geology, Environmental Science, Meteorology and other scientific areas can be interpreted by random fields. As a random field we refer to a family of random variables indexed over a domain which may represent space, time or any other dependence of interest. Random fields indexed over the sphere are extremely well studied in Probability and Spatial Statistics, because of their strong connection with CMB (Cosmic Microwave Background Radiation) and other phenomena in the above mentioned research fields; see indicatively [2, 5, 7, 10, 13, 16, 19, 27, 33, 36]. Today Probability and Statistics community has well-establish the corresponding theory on the sphere, hypersphere and more general manifolds.

Moreover, the exploration of more than one phenomena jointly distributed and correlated, can be aligned with the study of multivariate random fields. For some relevant literature we refer to [2, 28, 29, 30].

Going one step ahead, applications for instance in Oncology, Embryology, Geology, Mineralogy and Seismology may demand the exploration of the interior of the sphere —the ball— rather than the spherical surface itself. The need for studying random fields on the ball is straightforward, in the perspective of both applications and theory, and it is commonly expected that in the next few years significant attention of the community will be focused on this framework.

Random fields on the *ball* have just very recently been considered and drastic progress has been made in the two papers [25, 26]. To be more precise, the covariance matrix of such a m-variate isotropic random field, depends only on the distance between the index points lying on the ball, via a covariance kernel matrix-valued function. In [25] general series' forms of random fields have been considered and expression of their covariance kernels has been extracted. we will see more details in Section 2. In [26] we can find a number of developments on the spectral theory of random fields on the ball, together with future research directions in the area. As it has already been observed, the study on the ball may be substantially more involved than on the sphere or other homogeneous spaces and demands careful handling [21, 37].

The research trajectory that has been drawn in [24] on the sphere and in [20] on more general manifolds, is the expression of the regularity and continuity of random fields in

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terms of their covariance kernels. Moreover, it ends up that all the stochastic information is contained in the spectrum coefficients of the random field; with the prevailing terminology (coming from the Astrophysics) as the "angular power spectrum". Nonparametric conditions in the spectrum determine the regularity and the existence of modifications of the random field which are (more than) continuous. These two comprehensive articles have deeply inspired the communities of Probability and Statistics; see for instance [2, 3, 6, 8, 11, 12, 13, 14, 22] and the references therein.

In the *present paper* we consider m-variate isotropic random fields on the ball and explore a number of their properties such as regularity, continuity and modifications. All these problems are addressed here for the first time in the multivariate set-up. Moreover, our conditions are nonparametric and can cover under their umbrella a large class of random fields. Furthermore, we approximate the random fields by sequences of random fields suitable for simulations, obtaining sharp errors. Our study is supported by simulations illustrating our outcomes.

The study of random fields in the ball is very young and promising. Here we explore problems in the spirit of [20, 24], while we expand some reasonable research directions at the end of the paper.

Let us summarize the *contributions* of our paper:

(α) We study the regularity of the covariance matrix kernel of m-variate isotropic random fields, in terms of Sobolev spaces adapted to matrix-valued functions (Theorem 3.1). Precisely we express the Sobolev norm of the covariance kernel via the weighted ℓ^2-norm of their spectrum matrix-sequence.

(β) We illustrate the notion of regularity by numerical experiments in Section 4.

(γ) We provide ℓ^1-conditions on the spectrum matrix-sequence of m-variate isotropic random fields, which guarantee the Hölder continuity of their covariance kernel (Theorem 5.1). We compare the findings with the Sobolev regularity in Corollary 5.2.

(δ) We study the sample Hölder continuity of Gaussian random fields (see Theorem 6.1) and we derive a Kolmogorov-Chentsov-type result (Theorem 6.3).

(ε) We approximate m-variate isotropic random fields by truncations of their series' representation, under ℓ^∞-nonparametric assumptions. We estimate the error both in the sense of mixed-Lebesgue norms and $\mathbb{P}$-almost surely (Theorem 7.1).

Note further that our study of regularity and continuity properties, as in (α), (γ) and (δ) is the first of its kind on the multivariate framework. To this end we prove such theorems for multivariate random fields on the sphere (see Theorems 8.1 and 8.2). Furthermore, all our results are new for the case of univariate random fields on the ball.

The plan of the paper is as follows. In Section 2, we revise all the necessary background on multivariate random fields on the ball. Section 3 is dedicated to the study of the regularity properties of the random field via its covariance kernel. The findings are supported by simulation experiments placed in Section 4. The continuity study appears in Section 5. Modifications of multivariate random fields is the objective of the sixth Section, while in the seventh Section, we approximate random fields by truncated expansions and estimate the error. Regularity and continuity properties of spherical random fields can be found in Section 8. Section 9 contains overall discussions on this paper and future directions. All the proofs and some information about simulations are placed in the Appendix.

Notation. Let us fix here some necessary notation. We denote by $\mathbb{N}, \mathbb{N}_0$ the sets of positive and non-negative integers respectively and by $\mathbb{R}$ the space of real numbers. Let $m \in \mathbb{N}$. The space $\mathbb{R}^m$ will contain all the m-dimension vectors and the space $\mathbb{R}^{m\times m}$ all the $m \times m$ matrices with real entries. The identity of the last space is denoted by $\mathbf{I}_m$. Vectors and matrices will be denoted by bold lower case and capital letters respectively. The notation $a \sim b$ will indicate that there exists a constant $c \geq 1$ such that $c^{-1}b \leq a \leq cb$. By c we will denote a positive number who is independent of the main parameters in an equation and may vary from appearance to another. When we want to highlight that it depends on some precise parameter q, we will state it as c_q. Finally, for every real number x we denote by

$$(x)_+ := \max(x, 0). \tag{1.1}$$

2. Random fields on the Ball

Let $d \in \mathbb{N}$ such that $d \geq 2$ and let us denote by

$$\mathbb{B}^d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq 1\} \quad \text{the unit ball of } \mathbb{R}^d \quad \text{and}$$

$$\mathbb{S}^d = \{\mathbf{y} \in \mathbb{R}^{d+1} : \|\mathbf{y}\| = 1\} \quad \text{the unit hypersphere of } \mathbb{R}^{d+1}.$$

Then every point $\mathbf{x} \in \mathbb{B}^d$ can be uniquely mapped to

$$\mathbf{y} = \mathbf{y}(\mathbf{x}) = \big(\mathbf{x}, \sqrt{1 - \|\mathbf{x}\|^2}\big) \in \mathbb{S}^d_+,$$

where $\mathbb{S}^d_+ = \{\mathbf{y} = (y_1, \ldots, y_{d+1}) \in \mathbb{S}^d : y_{d+1} \geq 0\}$ the upper hypersphere of $\mathbb{R}^{d+1}$.

The *distance* $\rho(\mathbf{x}_1, \mathbf{x}_2)$ between two points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d$ is defined as the spherical distance between their images $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{S}^d_+$, $\varrho_{\mathbb{S}^d}(\mathbf{y}_1, \mathbf{y}_2)$. Precisely

$$\rho(\mathbf{x}_1, \mathbf{x}_2) := \arccos\big(\mathbf{x}_1'\mathbf{x}_2 + \sqrt{1 - \|\mathbf{x}_1\|^2}\sqrt{1 - \|\mathbf{x}_2\|^2}\big) \tag{2.2}$$

$$= \arccos(\mathbf{y}_1'\mathbf{y}_2) =: \varrho_{\mathbb{S}^d}(\mathbf{y}_1, \mathbf{y}_2). \tag{2.3}$$

Note that the above distance is invariant under orthogonal matrix transformations.

A m-variate random field $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$, is a collection of random vectors of $\mathbb{R}^m$, indexed over the ball $\mathbb{B}^d$. Precisely let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbf{Z} : \mathbb{B}^d \times \Omega \to \mathbb{R}^m$ a vector valued function. Then for every $\mathbf{x} \in \mathbb{B}^d$ and $\omega \in \Omega$,

$$\mathbf{Z}(\mathbf{x}, \omega) = \big(Z_1(\mathbf{x}, \omega), \ldots, Z_m(\mathbf{x}, \omega)\big)'$$

is a random vector.

The random field $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$, is called *isotropic* when it has finite second-order moments, zero mean;

$$\mathbb{E}\big(\mathbf{Z}(\mathbf{x})\big) = \mathbf{0}, \quad \text{for every } \mathbf{x} \in \mathbb{B}^d$$

and its covariance matrix function depends only on the distance $\rho(\mathbf{x}_1, \mathbf{x}_2)$, between $\mathbf{x}_1$ and $\mathbf{x}_2$. This means that there exist matrix-valued functions $\mathbf{C}_0 : [0, \pi] \to \mathbb{R}^{m\times m}$ and $\mathbf{C} : [-1, 1] \to \mathbb{R}^{m\times m}$, such that

$$\begin{aligned} \operatorname{cov}\big(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(\mathbf{x}_2)\big) &=: \mathbf{C}_0\big(\rho(\mathbf{x}_1, \mathbf{x}_2)\big) \\ &=: \mathbf{C}\big(\cos\big(\rho(\mathbf{x}_1, \mathbf{x}_2)\big)\big), \quad \text{for every } \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d. \end{aligned} \tag{2.4}$$

The matrix-valued function $\mathbf{C}$ will be referred as the *covariance kernel* of the isotropic random field $\mathbf{Z}$ and we will see that it is of fundamental importance for the study of its properties. Of course the covariance of an isotropic m-variate random field on the ball, is again invariant under transformations by orthogonal matrices. Note that the zero-mean assumption can be equally replaced by a constant mean and has been considered just for simplicity.

The random field $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$ will be referred as *mean square continuous* when for every $k = 1, \ldots, m$,

$$\mathbb{E}\big(\big|Z_k(\mathbf{x}_1) - Z_k(\mathbf{x}_2)\big|^2\big) \to 0, \quad \text{when } \rho(\mathbf{x}_1, \mathbf{x}_2) \to 0. \tag{2.5}$$

2.1. **Gegenbauer polynomials.** It is well known that orthogonal polynomials play a key role in the study of random fields. As a standard reference, see [35]. Let $\lambda > 0$. The Gegenbauer polynomials $\{P_n^{(\lambda)}(u) : n \in \mathbb{N}_0\}$ are the coefficients of w^n of the power series expansion of

$$(1 - 2uw + w^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(u) w^n, \quad |w| < 1,\ u \in \mathbb{R}. \tag{2.6}$$

For $\lambda = 0$ we define $P_n^{(0)}(\cos\theta) := \cos(n\theta)$, $n \in \mathbb{N}_0$ (the Chebyshev polynomials). There are strong similarities between the cases $\lambda > 0$ and $\lambda = 0$, but also some differences. We have unified everything in Section 8 of [3] and we summarize below the properties necessary for our study.

These polynomials form an orthogonal basis for the space of the square integrable functions on $[-1, 1]$, with respect to the weight $(1 - u^2)^{\lambda - \frac{1}{2}}$. Precisely

$$\int_{-1}^{1} P_n^{(\lambda)}(u) P_{n'}^{(\lambda)}(u)(1 - u^2)^{\lambda - \frac{1}{2}} du \sim \delta_{nn'} (n + 1)^{\mathfrak{t}(\lambda)} \tag{2.7}$$

$$\text{where } \mathfrak{t}(\lambda) = \begin{cases} 2\lambda - 2, \lambda > 0 \\ 0, \qquad \lambda = 0 \end{cases} \tag{2.8}$$

where δ is the classical Kronecker's delta. For every $n \in \mathbb{N}_0$, the function $P_n^{(\lambda)}$ is a polynomial of degree n. Let us list some fundamental properties of these polynomials below.

The polynomials (restricted to $[-1, 1]$), *maximize* in $u = 1$ and behave as follows:

$$\max_{u \in [-1,1]} \left|P_n^{(\lambda)}(u)\right| = P_n^{(\lambda)}(1) \sim (n + 1)^{(2\lambda - 1)_+}, \tag{2.9}$$

recall the notation in (1.1).

Let $\nu \in \mathbb{N}$. The behaviour of their *derivatives* is as follows:

$$\frac{d^\nu}{du^\nu} P_n^{(\lambda)}(u) = \eta(\lambda, n, \nu) P_{n-\nu}^{(\lambda+\nu)}(u), \quad \text{for every } n \geq \nu, \tag{2.10}$$

where the auxiliary function η above behaves as below (see [3] for the details)

$$\eta(\lambda, n, \nu) \sim \begin{cases} (n + 1),\ \lambda = 0 \\ 1, \qquad \lambda > 0 \end{cases}, \quad \text{for every } n \geq \nu \geq 0. \tag{2.11}$$

The last combined with (2.7) implies that the derivatives $\left\{\frac{d^\nu}{du^\nu} P_n^{(\lambda)}\right\}_{n \geq \nu}$ form an orthogonal basis of the square integrable functions in $[-1, 1]$ equipped with the weight $(1 + u^2)^{\lambda + \nu - \frac{1}{2}}$.

Finally Funk-Hecke formula asserts that for every $n, k \in \mathbb{N}_0$ and $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{S}^d$:

$$\int_{\mathbb{S}^d} P_n^{(\frac{d-1}{2})}(\mathbf{y}_1'\mathbf{u}) P_k^{(\frac{d-1}{2})}(\mathbf{y}_2'\mathbf{u}) d\mathbf{u} = \delta_{nk} \frac{|\mathbb{S}^d|}{a_n^2} P_n^{(\frac{d-1}{2})}(\mathbf{y}_1'\mathbf{y}_2), \tag{2.12}$$

where $|\mathbb{S}^d|$ the total area of the sphere $\mathbb{S}^d$ and

$$a_n^2 = \frac{2n + d - 1}{d - 1}, \quad n \in \mathbb{N}_0. \tag{2.13}$$

2.2. **Series representation.** In [25, Theorem 1] Lu, Leonenko and Ma, study m-variate random fields on the ball, satisfying a series' representation as in (2.16). They prove that these random fields are isotropic and provide the corresponding expression for their covariance kernel. Precisely:

Let $\{\mathbf{V}_n : n \in \mathbb{N}_0\}$ be a sequence of independent m-variate random vectors with zero mean and covariance given by

$$\operatorname{cov}(\mathbf{V}_n, \mathbf{V}_n) = a_n^2 \mathbf{I}_m. \tag{2.14}$$

Let $\mathbf{U}$ be a $(d+1)$-valued random vector uniformly distributed on the hypersphere $\mathbb{S}^d$, which is independent of $\{\mathbf{V}_n\}_n$.

Let $\{\mathbf{B}_n : n \in \mathbb{N}_0\}$ a sequence of positive definite matrices on $\mathbb{R}^{m\times m}$ such that the series of matrices

$$\sum_{n=0}^{\infty} \mathbf{B}_n P_n^{(\frac{d-1}{2})}(1) \quad \text{to be convergent on} \quad \mathbb{R}^{m\times m}. \tag{2.15}$$

Then the m-variate random field

$$\mathbf{Z}(\mathbf{x}) = \sum_{n=0}^{\infty} \mathbf{B}_n^{\frac{1}{2}} \mathbf{V}_n P_n^{(\frac{d-1}{2})}\big(\cos\big(\varrho_{\mathbb{S}^d}(\mathbf{y}, \mathbf{U})\big)\big), \quad \mathbf{x} \in \mathbb{B}^d, \text{ where } \mathbf{y} = \mathbf{y}(\mathbf{x}), \tag{2.16}$$

has zero mean, is isotropic on $\mathbb{B}^d$ and its covariance kernel with the notation as in (2.4) is given by

$$\mathbf{C}(u) = \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{(\frac{d-1}{2})}(u), \quad u \in [-1, 1]. \tag{2.17}$$

Note that above, $\mathbf{B}_n^{\frac{1}{2}}$ is the matrix of $\mathbb{R}^{m\times m}$ such that $\mathbf{B}_n^{\frac{1}{2}}\big(\mathbf{B}_n^{\frac{1}{2}}\big)' = \mathbf{B}_n$, which exists by the positive definitness of $\mathbf{B}_n$. The convergence in (2.15) should be understood in the coordinate wise sense and it is equivalent with

$$\sum_{n=0}^{\infty} \mathbf{B}_n (n+1)^{(d-2)_+} \quad \text{converges on} \quad \mathbb{R}^{m\times m}, \tag{2.18}$$

in the light of (2.9) and with the notation as in (1.1). It further guarantees the uniform convergence of $\mathbf{C}$ by a matrix-valued version of M-Weierstrass test and (2.9), (2.17).

For "increasing the stochasticity" of the m-variate random field, we use the following variant of the above.

Proposition 2.1. *Let $\{\mathbf{V}_n\}_n$ and $\{\mathbf{B}_n\}_n$ as above. Let also $\{\mathbf{U}_n : n \in \mathbb{N}_0\}$ be sequence of independent $(d+1)$-valued random vectors uniformly distributed on the hypersphere $\mathbb{S}^d$, which are independent of $\{\mathbf{V}_k\}_k$. Then the random field*

$$\mathbf{Z}(\mathbf{x}) = \sum_{n=0}^{\infty} \mathbf{B}_n^{\frac{1}{2}} \mathbf{V}_n P_n^{(\frac{d-1}{2})}\big(\cos\big(\varrho_{\mathbb{S}^d}(\mathbf{y}, \mathbf{U}_n)\big)\big), \quad \mathbf{x} \in \mathbb{B}^d, \text{ where } \mathbf{y} = \mathbf{y}(\mathbf{x}), \tag{2.19}$$

has zero mean, is isotropic on $\mathbb{B}^d$ and its covariance kernel as in (2.4) is given by (2.17).

The proof follows the lines of the one of [25, Theorem 1]; we skip the details.

For justifying the above variant we simulate univariate random fields on the ball $\mathbb{B}^3$ for a certain sequence $B_n = (n+1)^{-3}$ and normal V_n's. In both cases the ball $\mathbb{B}^3$ has been partially presented —just the first quarter of it— and then rotated for illustrating better the random phenomena. We will consider both cases during the current work, since both lead to the same covariance kernel (2.17).

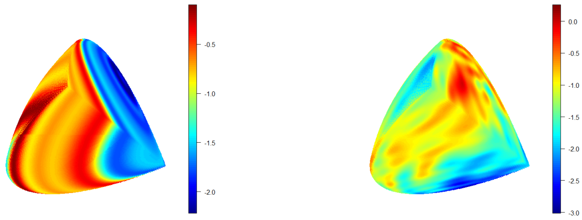


FIGURE 1. Left: simulated illustration of a random field as in (2.16). Right: simulated illustration of a random field as in (2.19).

2.3. **Covariance expression.** On the other hand, let $\{\mathbf{Z}(\mathbf{x})\}$ be a m-variate isotropic random field which is mean square continuous. Then by [25, Theorem 2], the covariance kernel as in (2.4), takes the form

$$\mathbf{C}(u) = \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{(\frac{d-2}{2})}(u), \quad u \in [-1,1], \tag{2.20}$$

where $(\mathbf{B}_n)$ a sequence of positive definite matrices such that

$$\sum_{n=0}^{\infty} \mathbf{B}_n P_n^{(\frac{d-2}{2})}(1) \quad \text{converges on} \quad \mathbb{R}^{m\times m}. \tag{2.21}$$

Remark 2.2. *A few remarks are in order.*

(1) *As the reader has already noted, the covariance kernels in (2.17) and (2.20), present a different index in the Gegenbauer polynomials. We refer to [25] for a discussion about it. We continue this discussion in Section 9.*
(2) *The convergence in (2.21) indicates the fundamental importance of the behaviour of the matrix-sequence* $(\mathbf{B}_n)$.

Let us illustrate the notion of isotropy. The covariance $\mathrm{cov}(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(\mathbf{x}_2))$ depends only on the distance between the points $\mathbf{x}_1$ and $\mathbf{x}_2$ via the kernel $\mathbf{C}$ as in (2.4) and this is what we aim to present in the next plots. Consider a univariate isotropic random field $\{Z(\mathbf{x})\}$ on the ball $\mathbb{B}^3$ with $B_n = (n+1)^{-3}$, visualized in Section 2.2. We fix $\mathbf{x}_2 = \mathbf{0}$ and *we draw the covariance* $\mathrm{cov}(Z(\mathbf{x}), Z(\mathbf{0})) : \mathbb{B}^3 \to \mathbb{R}$: where on the left we present the

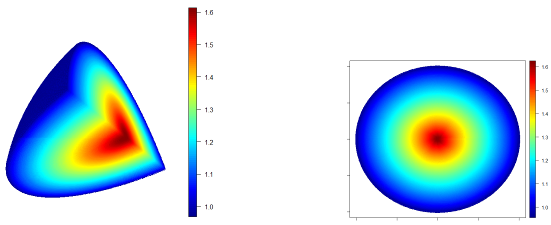


FIGURE 2. Illustration of isotropy.

first quarter of the ball rotated as before and on the right we visualize the covariance for $x_3 = 0$. Note that the plots are obtained by truncating the sequence in (2.17) using the terms from $n = 0$ to $n = 100$. Simple arguments can show that the true covariance as in (2.17) differs from the truncation presented above not more than 10^{-4}.

3. Regularity

In this section we will explore the regularity properties of m-variate isotropic random fields on the ball. We need a proper definition of regularity first. The notion of regularity use to be expressed in terms of the norms of regularity (or smoothness) spaces; Sobolev, Besov spaces, etc. The classical approach is to study the regularity of the covariance function (or kernel). It has been recently spotted that this is completely accurate [20] (at least for univariate random fields).

For the case of the sphere, regularity has been studied in detail in [24] and pioneered the progress in the area. For random fields on more general manifolds and products of manifolds see for example [12, 13].

Although, there is not any relevant result on the ball. Moreover, there is not any similar progress in the multivariate set-up. Our first result is dedicated to this two-fold purpose; study the regularity properties on random fields which are: (i) multivariate and (ii) indexed on the ball.

The multivariate setting demands some careful adaptation of the notions of integrability and regularity. We will see some Lebesgue and Sobolev spaces for matrix-valued functions. We present here the following definitions.

Let $\mathbf{A}$ and $\mathbf{B}$ be two $\mathbb{R}^{m\times k}$ matrices, $m,k\in\mathbb{N}$. Their *Frobenius inner product* is defined as

$$\langle \mathbf{A},\mathbf{B}\rangle_F = \operatorname{trace}\big(\mathbf{A}\mathbf{B}'\big). \tag{3.22}$$

The corresponding *Frobenius norm* $\|\mathbf{A}\|_F$ of $\mathbf{A}$ takes the form

$$\|\mathbf{A}\|_F^2 = \langle \mathbf{A},\mathbf{A}\rangle_F = \sum_{j=1}^{k}\sum_{i=1}^{m}(\mathbf{A}_{ij})^2. \tag{3.23}$$

Let $\lambda\geq 0$ and $m\in\mathbb{N}$. We say that the matrix valued function $\mathbf{C}:[-1,1]\to\mathbb{R}^{m\times m}$ belongs to the (weighted) *Lebesgue space* $L^2_{(\lambda;m)}$ when

$$\big\|\mathbf{C}\big\|^2_{L^2_{(\lambda;m)}} = \int_{-1}^{1}\big\|\mathbf{C}(u)\big\|_F^2(1-u^2)^{\lambda-\frac{1}{2}}\,du<\infty. \tag{3.24}$$

Let also $s\in\mathbb{N}_0$. We define the *Sobolev space* $W^s_{(\lambda;m)}$ as the set of all matrix-valued functions $\mathbf{C}:[-1,1]\to\mathbb{R}^{m\times m}$, such that

$$\big\|\mathbf{C}\big\|^2_{W^s_{(\lambda;m)}} = \sum_{\nu=0}^{s}\big\|\mathbf{C}^{(\nu)}\big\|^2_{L^2_{(\lambda+\nu;m)}}<\infty. \tag{3.25}$$

Above $\mathbf{C}^{(\nu)}$ should be considered in the coordinate-wise sense. Note that the above Sobolev spaces are the natural matrix-valued version of the spaces used in [37]. To the best of our knowledge, this is the first time such matrix-valued regularity spaces are used for the study of random fields.

Finally, a simple but important property is the following

$$L^2_{(\lambda;m)} = W^0_{(\lambda;m)}\supset W^1_{(\lambda;m)}\supset W^2_{(\lambda;m)}\supset\cdots \tag{3.26}$$

We are ready to present our regularity results. We start by a general Theorem:

Theorem 3.1. *Let $\lambda\geq 0$, $s\in\mathbb{N}_0$ and a matrix-valued function $\mathbf{C}:[-1,1]\to\mathbb{R}^{m\times m}$ of the form*

$$\mathbf{C}(u) = \sum_{n=0}^{\infty}\mathbf{B}_n P_n^{(\lambda)}(u),\quad u\in[-1,1], \tag{3.27}$$

where $(\mathbf{B}_n)$ *a sequence of positive definite matrices such that*

$$\sum_{n=0}^{\infty} \mathbf{B}_n P_n^{(\lambda)}(1) \quad \text{converges on} \quad \mathbb{R}^{m\times m}. \tag{3.28}$$

Then $\mathbf{C}$ *belongs to the Sobolev space* $W^s_{(\lambda;m)}$ *if and only if*

$$\sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2s+\mathfrak{t}(\lambda)} < \infty. \tag{3.29}$$

Precisely,

$$\|\mathbf{C}\|^2_{W^s_{(\lambda;m)}} \sim \sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2s+\mathfrak{t}(\lambda)}. \tag{3.30}$$

Note that the auxiliary function $\mathfrak{t}(\lambda)$ has been presented in (2.8).
The proof of Theorem 3.1, as every other proof, has been delivered to the Appendix for streamlining the exposition of our study. Simulation studies illustrating the notion of regularity can be found in Section 4. We turn now to translate the general regularity Theorem in the language of random fields *on the ball*.

Corollary 3.2. *Let* $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$ *be a* m*-variate isotropic mean square continuous random field on the ball* $\mathbb{B}^d$ *and* $s \in \mathbb{N}_0$*. Then its covariance kernel* $\mathbf{C}$*, expressed as in (2.20), belongs to the Sobolev space* $W^s_{(\frac{d-2}{2};m)}$ *if and only if*

$$\sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2s+\mathfrak{t}(\frac{d-2}{2})} < \infty. \tag{3.31}$$

Precisely,

$$\|\mathbf{C}\|^2_{W^s_{(\frac{d-2}{2};m)}} \sim \sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2s+\mathfrak{t}(\frac{d-2}{2})} \tag{3.32}$$

For the explicit series expansions of random fields in (2.16) and (2.19), the regularity is summarized in the following:

Corollary 3.3. *Let* $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$ *be a* m*-variate random field on the ball* $\mathbb{B}^d$*, as in (2.16) or (2.19) and let* $s \in \mathbb{N}_0$*. Then its covariance kernel* $\mathbf{C}$*, in (2.17), belongs to the Sobolev space* $W^s_{(\frac{d-1}{2};m)}$ *if and only if*

$$\sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{d+2s-3} < \infty. \tag{3.33}$$

Precisely,

$$\|\mathbf{C}\|^2_{W^s_{(\frac{d-1}{2};m)}} \sim \sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{d+2s-3} \tag{3.34}$$

Note that Theorem 3.1 in the abstract formed expressed above can be used for the study of multivariate random fields on the sphere, too. For streamlining our presentation we have placed the study of the sphere in Section 8.

4. Simulation Study

Let us support our regularity results by some simulations. We will use the series representations (2.19):

$$\mathbf{Z}(\mathbf{x}) = \sum_{n=0}^{\infty} \mathbf{B}_n^{\frac{1}{2}} \mathbf{V}_n P_n^{(\lambda)}(\mathbf{y}'\mathbf{U}_n), \quad \lambda = \frac{d-1}{2}, \ \ \mathbf{y} = \mathbf{y}(\mathbf{x}).$$

Assume that the spectrum sequence $(\mathbf{B}_n)$ is such that

$$\|\mathbf{B}_n\|_F \sim (n+1)^{-\alpha}, \quad \text{for some } \alpha > 0. \tag{4.35}$$

For having a proper covariance kernel we need $\alpha > d-1$; see (2.21).

Let $s \in \mathbb{N}_0$. By Theorem 3.3 the corresponding covariance kernel

$$\mathbf{C} \in W^s_{(\frac{d-1}{2};m)} \Leftrightarrow s < \alpha + 1 - \frac{d}{2}. \tag{4.36}$$

We will illustrate random fields enjoying different regularity levels on the unit disk $\mathbb{B}^2$ and the unit ball $\mathbb{B}^3$ in the univariate and bivariate case ($m \in \{1, 2\}$). Technical details appear in the Appendix.

We simulate first scalar random fields; $m = 1$.

We start by the case of the unit disk $\mathbb{B}^2$ ($d = 2$). Then $\frac{d-1}{2} = \frac{1}{2}$ and the Gegenbauer polynomials correspond to Legendre ones. In our experiments the random variables V_n are assumed to be Gaussian with $V_n \sim \mathcal{N}(0, 2n+1)$; see (2.13).

We consider the (univariate) random fields Z_j, $j = 1, \dots, 4$, of the form (2.19), with $B_n^j = (n+1)^{-\alpha_j}$ and take $\alpha_j = 1+j$ for every $j = 1, \dots, 4$. Then their covariance kernels C_j belong to $W^j \setminus W^{j+1}$ respectively (where $W^s = W^s_{(\frac{1}{2};1)}$). We visualize Z_j's in Figure 4 and the difference on the regularity level becomes apparent:

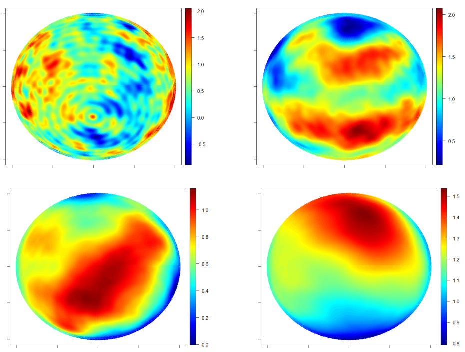


Figure 3. Z_j, $j = 1, 2, 3, 4$, from the top left to the bottom right.

We proceed to the case of the unit ball $\mathbb{B}^3$ ($d = 3$). Then $\frac{d-1}{2} = 1$ and $\mathrm{Var}(V_n) = n+1$; see (2.13).

We consider the (univariate) random fields Z_j, $j = 1, \dots, 4$, of the form (2.19), with $V_n \sim \mathcal{N}(0, n+1)$, $B_n^j = (n+1)^{-\alpha_j}$ and take $\alpha_j = 2 + j$, $j = 1, \dots, 4$.

Then their covariance kernels C_j belong to $W^{j+1} \setminus W^{j+2}$ respectively (where $W^s = W^s_{(1;1)}$). We visualize Z_j's in Figure 4.

We now turn to the main interest of this study which is the case of multivariate random fields. It suffices to focus on the bivariate case; $m = 2$. Consider $\mathbf{Z} = (Z_1, Z_2)'$ a bivariate isotropic random field on $\mathbb{B}^d$ expressed as in (2.19). Let $r = \mathrm{cor}(Z_1, Z_2)$. We will illustrate such random fields for several values of regularity level and correlation

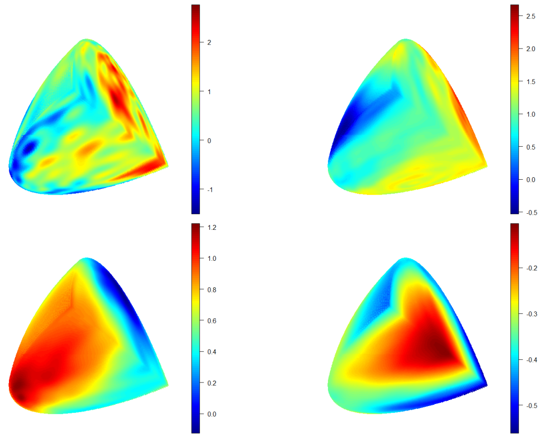


FIGURE 4. Note that we have truncated the unit ball presenting the first quarter of it and then we rotated for better illustration. Moving from top left to bottom right $(Z_1, \dots, Z_4)$, we can observe the increase of the level of regularity that the kernels enjoy.

between the coordinate random fields on $\mathbb{B}^2$ and $\mathbb{B}^3$. Precise details are available in Section 10.

Let us study first the case of the disk $\mathbb{B}^2$. We provide initially random fields whose covariance enjoys the membership in $W^1_{(\frac{1}{2};2)}$ in Figure 4. In Figure 4 we illustrate smoother

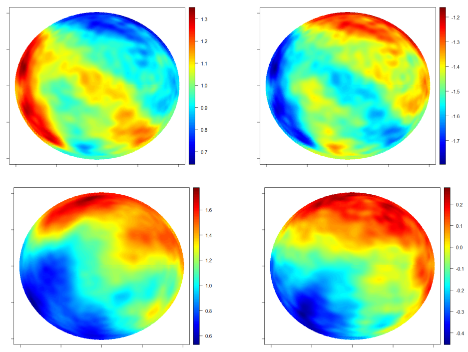


FIGURE 5. In the first row we have very high negative correlation (here $r = -0.9$) and in the second, high positive correlation (here $r = 0.6$).

random fields on $W^2_{(\frac{1}{2};2)}$.

We proceed to the case of the ball $\mathbb{B}^3$. We draw random fields on $W^2_{(1;2)}$ in Figure 4 and on $W^4_{(1;1)}$ in Figure 4.

5. Hölder continuity

In this section we explore the Hölder continuity of isotropic mean square continuous m-variate random fields, expressed in terms of their covariance kernels. Hölder continuity is a way to measure the variation of the function in terms of the variation of its variable. A stronger property than continuity. Here we study the Hölder continuity in the covariance kernel and in the next section we transfer this to the original random field; Theorem 6.1.

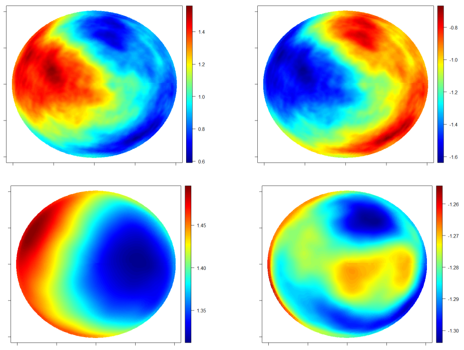


FIGURE 6. First row: very high negative correlation (here $r = -0.9$). Second row: very low positive correlation (here $r = 0.28$).

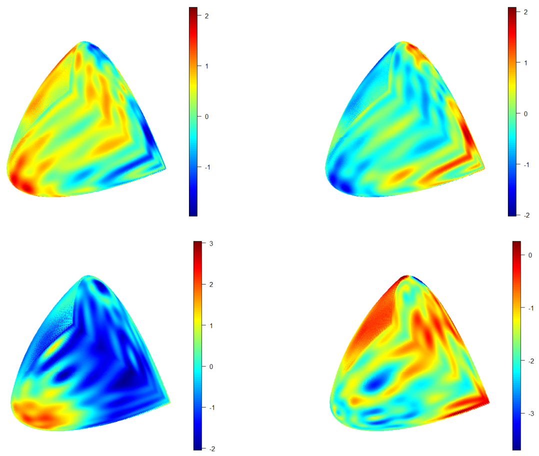


FIGURE 7. Random fields on $W^2_{(1;2)}$. First row: very high negative correlation (here $r = -0.9$) and second row: high negative correlation ($r = -0.6$).

Our main aim is to obtain a Kolmogorov-Chentsov-type Theorem, (see Theorem 6.3). Such an approach for univariate random fields appears in [24] and, later, in [12].

We need first to determine a way to deal with the continuity of matrix-valued functions.

Let $0 < \delta \leq 1$. We say that a matrix-valued function $\mathbf{C} : [-1, 1] \to \mathbb{R}^{m\times m}$ is *δ-Hölder continuous* when

$$\left\|\mathbf{C}\right\|_{\mathcal{H}^\delta_m} = \sup_{u\in[-1,1]} \left\|\mathbf{C}(u)\right\|_F + \sup_{u\neq t} \frac{\left\|\mathbf{C}(u) - \mathbf{C}(t)\right\|_F}{|u-t|^\delta} < \infty. \tag{5.37}$$

We are now in place to obtain the Hölder continuity of the covariance kernel of an isotropic m-variate random field, based on the (weighted) summability of its coefficient matrices $(\mathbf{B}_n)$. Precisely we prove the following:

Theorem 5.1. *Let $\left\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\right\}$ be a m-variate mean square continuous isotropic random field on the ball $\mathbb{B}^d$ and $0 < \delta \leq 1$. Then its covariance kernel $\mathbf{C}$, expressed as*

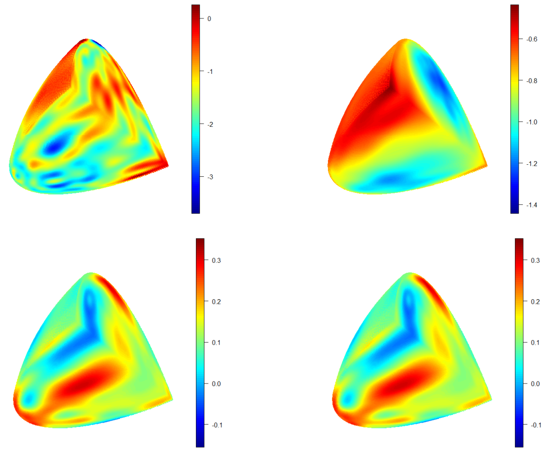


FIGURE 8. Random fields on $W^4_{(1;2)}$. First row: very low (positive) correlation (here $r = 0.28$) and second row: high positive correlation (here $r = 0.9$).

in (2.20), is δ-Hölder continuous provided that

$$\sum_{n=0}^{\infty} \left\| \mathbf{B}_n \right\|_F (n+1)^{2\delta+(d-3)_+} < \infty. \tag{5.38}$$

Moreover

$$\left\| \mathbf{C} \right\|_{\mathcal{H}^\delta_m} \leq c \sum_{n=0}^{\infty} \left\| \mathbf{B}_n \right\|_F (n+1)^{2\delta+(d-3)_+}. \tag{5.39}$$

The proof of Theorem 5.1 appears in the Appendix. In Section 6 we will use the above result for obtaining continuous modifications of multivariate isotropic random fields on the ball. A more general statement about covariances as in (3.27) could be proved, but we prefer to stick in the (special) case of our random fields.

5.1. **Discussion on the Hölder and Sobolev regularities.** A very natural question here is how related can be our results in Sobolev regularity and Hölder continuity. Note that Theorem 3.1 is based on the (weighted) ℓ^2-behaviour of the coefficients $(\mathbf{B}_n)$ and Theorem 5.1 on the (weighted) ℓ^1-behaviour of the same matrix-sequence. We express the answer in the next corollary:

Corollary 5.2. *Let $\left\{ \mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d \right\}$ be a m-variate mean square continuous isotropic random field on the ball $\mathbb{B}^d$ and let $\mathbf{C}$ be its covariance kernel expressed as in (2.20).*

(a) Let $s \in \mathbb{N}_0$ with $s > \frac{d-1}{2}$ and assume that $\mathbf{C}$ is Sobolev regular of order s. Then $\mathbf{C}$, is δ-Hölder continuous for every $0 < \delta \leq 1$, such that $0 < \delta < \frac{1}{2}\left(s - \frac{d-1}{2}\right)$.

(b) Let $0 < \delta \leq 1$ and assume that $\mathbf{C}$ enjoys (5.38). Then $\mathbf{C}$ is Sobolev regular of order s, for every $s \in \mathbb{N}_0$ such that $s \leq 2\delta + \frac{d-2}{2}$.

6. MODIFICATIONS OF RANDOM FIELDS

In this Section we transfer the continuity study to the m-variate random field itself. The main purpose of the Section is to arrive in a Kolmogorov-Chentsov-type Theorem; a sufficient condition for the existence of Hölder continuous modifications of multivariate random fields on the ball.

We start by giving some necessary definitions adapted to m-variate random fields on the ball.

Let $0 < \gamma \leq 1$. We say that an m-variate random field $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$ is *γ-sample Hölder continuous* on the ball, when for every $\omega \in \Omega$, the vector-valued sample function $\mathbf{Z}_\omega(\mathbf{x}) = \mathbf{Z}(\mathbf{x}, \omega)$, $\mathbf{x} \in \mathbb{B}^d$, is γ-Hölder continuous in the sense that: There exists a constant $0 \leq c < \infty$ such that

$$\left\|\mathbf{Z}(\mathbf{x}, \omega)\right\|_F \leq c \text{ and } \left\|\mathbf{Z}(\mathbf{x}_1, \omega) - \mathbf{Z}(\mathbf{x}_2, \omega)\right\|_F \leq c\rho(\mathbf{x}_1, \mathbf{x}_2)^\gamma, \tag{6.40}$$

for every $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d$.

We say that the m-variate random field $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$ is *locally γ-Hölder continuous*, when for every $\mathbf{x}_0 \in \mathbb{B}^d$, there exists a neighbourhood $V(\mathbf{x}_0)$ of $\mathbf{x}_0$, such that the restricted random field $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in V(\mathbf{x}_0)\}$ to be γ-sample Hölder continuous on $V(\mathbf{x}_0)$. That is to satisfy (6.40), for every $\omega \in \Omega$ and every $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in V(\mathbf{x}_0)$.

Finally we say that the m-variate random field $\{\tilde{\mathbf{Z}}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$ is a *modification* of $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$, when $\mathbb{P}(\tilde{\mathbf{Z}}(\mathbf{x}) = \mathbf{Z}(\mathbf{x})) = 1$, $\forall \mathbf{x} \in \mathbb{B}^d$.

We can now prove the next result which expresses Hölder continuity on the random field by itself. Here we need to assume that the random field is Gaussian.

Theorem 6.1. *Let $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$ be a Gaussian m-variate mean square continuous isotropic random field on the ball $\mathbb{B}^d$ and $0 < \delta \leq 1$. When the covariance kernel as in (2.20) satisfies (5.38), then for every $p \in \mathbb{N}$*

$$\mathbb{E}\Big(\left\|\mathbf{Z}(\mathbf{x}_1) - \mathbf{Z}(\mathbf{x}_2)\right\|_F^{2p}\Big) \leq c\rho(\mathbf{x}_1, \mathbf{x}_2)^{2\delta p}, \quad \textit{for every } \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d. \tag{6.41}$$

Remark 6.2. *On Theorem 6.1 we further assumed that $\mathbf{Z}$ is Gaussian. Technically, this was necessary because of the need of the moments in (10.73). Let us emphasize that the proof of such a result turns out to be much more involved than in the univariate case. For univariate random fields on a very general class of manifolds we refer to* [23].

The above result guarantees that small changes in the spatial domain, imply small changes in the expectations of the distance between the corresponding values of the original m-variate random field.

We are ready to prove a version of *Kolmogorov-Chentsov Theorem* for m-variate random fields on the ball. Precisely we will show the following:

Theorem 6.3. *Let $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$ be a Gaussian m-variate mean square continuous isotropic random field on the ball $\mathbb{B}^d$ and $0 < \delta \leq 1$. When the covariance kernel as in (2.20) satisfies (5.38), then there exists a modification $\{\tilde{\mathbf{Z}}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\}$ of $\mathbf{Z}$ which is locally γ-Hölder continuous, for every $0 < \gamma < \delta$.*

7. Approximation

We proceed to study truncated approximations of m-variate isotropic random fields. As we recalled from [25] and we saw in Proposition 2.1, the random fields

$$\mathbf{Z}(\mathbf{x}) = \sum_{n=0}^{\infty} \mathbf{B}_n^{\frac{1}{2}} \mathbf{V}_n P_n^{(\frac{d-1}{2})}(\mathbf{y}'\mathbf{U}), \quad \mathbf{y} = \mathbf{y}(\mathbf{x}),\ \mathbf{x} \in \mathbb{B}^d \quad \text{and} \tag{7.42}$$

$$\mathbf{Z}(\mathbf{x}) = \sum_{n=0}^{\infty} \mathbf{B}_n^{\frac{1}{2}} \mathbf{V}_n P_n^{(\frac{d-1}{2})}(\mathbf{y}'\mathbf{U}_n), \quad \mathbf{y} = \mathbf{y}(\mathbf{x}),\ \mathbf{x} \in \mathbb{B}^d, \tag{7.43}$$

are isotropic on $\mathbb{B}^d$, under certain assumptions on the above matrices and random vectors. Such expansions are suitable for truncation and have already been used successfully for simulations in Section 4. In the current Section we will rigorously justify the validity of truncated fast approximation.

Let $N \in \mathbb{N}$. We define the truncated random field $\mathbf{Z}_N$, of $\mathbf{Z}$ in (7.42), as

$$\mathbf{Z}_N(\mathbf{x}) = \sum_{n=0}^{N} \mathbf{B}_n^{\frac{1}{2}} \mathbf{V}_n P_n^{(\frac{d-1}{2})}\big(\mathbf{y}'\mathbf{U}\big), \quad \mathbf{y} = \mathbf{y}(\mathbf{x}),\ \mathbf{x} \in \mathbb{B}^d \tag{7.44}$$

and with the obvious modifications for the random field $\mathbf{Z}$ as in (7.43).

We will study the error of the natural approximation of $\mathbf{Z}$ by $(\mathbf{Z}_N)_N$ using mixed Lebesgue norms, respecting the nature of a random field as a spatial function and random variable.

Let $p > 0$. The $L^p(\mathbb{B}^d;\mathbb{R}^m)$-(quasi)norm of a vector-valued function $\mathbf{f} : \mathbb{B}^d \to \mathbb{R}^m$ is defined as

$$\|\mathbf{f}\|_{L^p(\mathbb{B}^d;\mathbb{R}^m)} = \Big(\int_{\mathbb{B}^d} \|\mathbf{f}(\mathbf{x})\|_F^p \mathrm{d}\mu(\mathbf{x})\Big)^{1/p}, \tag{7.45}$$

where $\mathrm{d}\mu(\mathbf{x})$ the measure element on $\mathbb{B}^d$.

The corresponding inner product of two vector-valued functions $\mathbf{f}, \mathbf{g} : \mathbb{B}^d \to \mathbb{R}^m$, takes the form:

$$\langle \mathbf{f}, \mathbf{g}\rangle_{L^2(\mathbb{B}^d;\mathbb{R}^m)} = \int_{\mathbb{B}^d} \langle \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})\rangle_F \mathrm{d}\mu(\mathbf{x}). \tag{7.46}$$

Let also $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The vector-valued function $\mathbf{f} : \mathbb{B}^d \times \Omega \to \mathbb{R}^m$ belongs to the mixed Lebesgue space $L^p\big(\Omega, L^q(\mathbb{B}^d;\mathbb{R}^m)\big)$, $p, q > 0$, if

$$\begin{aligned}\|\mathbf{f}\|_{L^p(\Omega,L^q(\mathbb{B}^d;\mathbb{R}^m))} &= \Big(\mathbb{E}\big(\|\mathbf{f}\|^p_{L^q(\mathbb{B}^d;\mathbb{R}^m)}\big)\Big)^{1/p} \\ &= \Big(\int_\Omega \|\mathbf{f}(\cdot,\omega)\|^p_{L^q(\mathbb{B}^d;\mathbb{R}^m)} \mathrm{d}\mathbb{P}(\omega)\Big)^{1/p} \\ &= \Bigg(\int_\Omega \Big(\int_{\mathbb{B}^d} \|\mathbf{f}(\mathbf{x},\omega)\|_F^q \mathrm{d}\mu(\mathbf{x})\Big)^{p/q} \mathrm{d}\mathbb{P}(\omega)\Bigg)^{1/p} < \infty. \end{aligned} \tag{7.47}$$

For the special case $p = q = 2$ and thanks to Fubini-Tonelli Theorem we have

$$\|\mathbf{f}\|^2_{L^2(\Omega,L^2(\mathbb{B}^d;\mathbb{R}^m))} = \mathbb{E}\big(\|\mathbf{f}\|^2_{L^2(\mathbb{B}^d;\mathbb{R}^m)}\big) = \int_{\mathbb{B}^d} \mathbb{E}\big(\|\mathbf{f}(\mathbf{x})\|_F^2\big)\mathrm{d}\mu(\mathbf{x}). \tag{7.48}$$

The approximation error estimation demands an ℓ^∞-condition for the spectrum matrix sequence of the random field. Our approximation result is as follows:

Theorem 7.1. *Let $\big\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\big\}$ be an isotropic random field on the ball $\mathbb{B}^d$ of the form (2.16) or (2.19). Let $\varepsilon > 0$ and assume that the sequence of matrices satisfies*

$$\mathit{trace}\big(\mathbf{B}_n\big) \le c_* n^{-d+1-\varepsilon}, \quad \text{for every } n \ge n_0, \tag{7.49}$$

for some $c_* > 0$ and some $n_0 \in \mathbb{N}$.

Then the corresponding series of truncated random fields $\big(\mathbf{Z}_N\big)_N$ converges to the random field $\mathbf{Z}$ in the following senses:

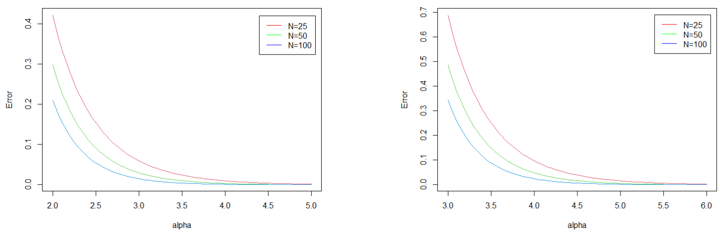
(a) In $L^p(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))$, for every $p > 0$ and there exists a constant $c > 0$ such that

$$\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^p(\Omega,L^2(\mathbb{B}^d;\mathbb{R}^m))} \le cN^{-\varepsilon/2}. \tag{7.50}$$

(b) $\mathbb{P}$-almost surely. Precisely the error is asymptotically bounded by

$$\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^2(\mathbb{B}^d;\mathbb{R}^m)} \le N^{-\alpha}, \quad \mathbb{P} - a.s., \tag{7.51}$$

for $\alpha < \varepsilon/2$.



Remark 7.2. *Some remarks are in order.*

(1) *The mixed Lebesgue (quasi-)norms used above are ideal for counting distance between random fields. These involve the expectation since we deal with random variables and count the total variability in the spatial domain using the standard measurement via Lebesgue (quasi-)norms.*

(2) *Let us point out here that $\|\mathbf{B}\|_F \sim trace(\mathbf{B})$, for every positive definite matrix $\mathbf{B}$. Therefore all Theorems 3.1, 5.1, 6.1 and 7.1 can be expressed equivalently in terms of $\|\mathbf{B}_n\|_F$ or $trace(\mathbf{B}_n)$.*

7.1. **Connection with regularity and simulations.** Let us connect the findings of this Section with those in Sections 3 and 4. We restrict to the $L^2(\Omega, L^2(\mathbb{B}^d; m))$. We consider the matrix sequence $\mathbf{B}_n$ to be as in Section 4, that is

$$\left\|\mathbf{B}_n\right\|_F = (n+1)^{-\alpha}, \quad \alpha > d-1. \tag{7.52}$$

As we saw in Section 4 the random field $\mathbf{Z}$, as in (2.16) or (2.19) with matrix sequence $(\mathbf{B}_n)$ as above enjoys membership in exactly all Sobolev spaces $W^s_{(\frac{d-1}{2};m)}$, with $s < \alpha + 1 - \frac{d}{2}$.

On the other hand: By Cauchy-Schwartz inequality we derive

$$\mathrm{trace}(\mathbf{B}_n) \le \sqrt{m} n^{-d+1-\varepsilon}, \quad \varepsilon = \alpha - (d-1) > 0. \tag{7.53}$$

A closer look in the proof of Theorem 7.1 asserts that the $L^2(\Omega, L^2(\mathbb{B}^d; m))$-error is bounded from above by

$$m^{1/4}\sqrt{(d-1)\mu(\mathbb{B}^d)}\frac{1}{\sqrt{\varepsilon N^{\varepsilon}}}. \tag{7.54}$$

From all the above we can extract that: When the regularity level grows, then the error decays exponentially. Consequently less and less terms (N) are needed for attaining a satisfactory error bound. In the plots below we present the decay of the error when α —and therefore the regularity— increases, for bivariate random field on $\mathbb{B}^2$ on the left and on $\mathbb{B}^3$ on the right. The smoothness on the horizontal axis spans from the space W^1 to W^4 (for the proper values of α). The different colors represent the truncation index $N \in \{25, 50, 100\}$.

8. Multivariate random fields on the sphere

Multivariate random fields on the sphere and more general compact manifolds have received recently significant amount of attention; see for example [2, 16, 29, 30, 32] and especially the works of Ma [28] and Ma and Malyarenko [31] for spheres and compact two-point homogeneous spaces respectively. Although, regularity properties have not yet been established for multivariate spherical random fields. Our abstract Theorem 3.1 will be translated to the regularity theorem for such multivariate random fields on the sphere. The same will happen with continuity Theorems 5.1, 6.1 and 6.3 too. For approximation results and simulations, see [2].

Let us first recall the following [28, 31]: Let $\{\mathbf{Z}(\mathbf{y}) : \mathbf{y} \in \mathbb{S}^d\}$ be a m-variate isotropic mean square continuous random field on the sphere $\mathbb{S}^d$. Then its covariance can be expressed as $\operatorname{cov}\big(\mathbf{Z}(\mathbf{y}_1), \mathbf{Z}(\mathbf{y}_2)\big) = \mathbf{C}\big(\mathbf{y}_1'\mathbf{y}_2\big)$, $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{S}^d$, for a covariance kernel matrix valued function of the form

$$\mathbf{C}(u) = \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{(\frac{d-1}{2})}(u), \quad u \in [-1, 1], \tag{8.55}$$

where $\{\mathbf{B}_n\}$ is a sequence of positive-definite $m \times m$ matrices such that the series $\sum \mathbf{B}_n P_n^{(\frac{d-1}{2})}(1)$ to be convergent.

The regularity of such random fields can be summarized in the following:

Theorem 8.1. *Let $\{\mathbf{Z}(\mathbf{y}) : \mathbf{y} \in \mathbb{S}^d\}$ be a m-variate isotropic mean square continuous random field on the sphere $\mathbb{S}^d$ and $s \in \mathbb{N}_0$. The covariance kernel $\mathbf{C}$ as in (8.55) belongs to the Sobolev space $W^s_{(\frac{d-1}{2};m)}$ if and only if*

$$\|\mathbf{C}\|^2_{W^s_{(\frac{d-1}{2};m)}} \sim \sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{d+2s-3} < \infty. \tag{8.56}$$

The proof of Theorem 8.1 is a direct application of the abstract Theorem 3.1; we skip further details.

The continuity and modification properties for spherical random fields yield by the weighted summability of $(\mathbf{B}_n)_n$. We modify properly the proofs of the results in Sections 5 and 6 and we summarize the outcome in the following:

Theorem 8.2. *Let $\{\mathbf{Z}(\mathbf{y}) : \mathbf{y} \in \mathbb{S}^d\}$ be a Gaussian m-variate isotropic mean square continuous random field on the sphere $\mathbb{S}^d$ and $0 < \delta \leq 1$. Assume that the covariance kernel $\mathbf{C}$ as in (8.55) satisfies*

$$\sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F (n+1)^{d+2\delta-2} < \infty. \tag{8.57}$$

Then

(a) The covariance kernel is δ-Hölder continuous; $\mathbf{C} \in \mathcal{H}_m^\delta$.

(b) For every $p \in \mathbb{N}$, there exists a constant $c = c_p > 0$ such that

$$\mathbb{E}\Big(\big\|\mathbf{Z}(\mathbf{y}_1) - \mathbf{Z}(\mathbf{y}_2)\big\|_F^{2p}\Big) \leq c\rho_{\mathbb{S}^d}(\mathbf{y}_1, \mathbf{y}_2)^{2\delta p}, \quad \textit{for every } \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{S}^d.$$

(c) There exists a modification $\big\{\tilde{\mathbf{Z}}(\mathbf{y}) : \mathbf{y} \in \mathbb{S}^d\big\}$ of $\mathbf{Z}$ which is locally γ-Hölder continuous, for every $0 < \gamma < \delta$.

9. Final remarks

Random fields on the ball are under-presented in the literature in comparison with the case of the sphere. This happens in many other scientific areas, because of the increasing difficulty that appears when studying the ball. The studies [25, 26] are the first two in the area and contain fruitful information.

In the present paper we covered several problems; regularity, simulations, continuity, modifications and approximations, while there are many natural directions for future research in the very area of our study and beyond.

For example Interpolation spaces and Hölder spaces for any order $\delta > 0$ can be used as in [24] and [12]. The simulation studies we performed are sufficient for the purpose of illustrating the notions of our interest. The study could be expanded by choosing more families of the spectrum sequence based in the results and examples in [25, 26]. The comprehensive study by Lang and Schwab [24] directly opens research directions, e.g. SPDEs, while Bayesian analysis' community may find some inspiration by [2, 9].

In the current framework a fundamental issue would be to find a constructive example of an isotropic random field on the ball, whose covariance agrees with the general expansion in (2.20). We continue this discussion from [25]. A random field as in (2.16) or (2.19) enjoys a covariance as in (2.4). Then using the formula (8) of Askey and Wainger [4], which goes back to Gegenbauer [18], we can arrive after some technical steps to the following expansion:

$$\mathbf{C}(u) = \sum_{r=0}^{\infty} \Big(\sum_{j=0}^{\infty} \mathbf{B}_{r+2j} a_j(r+2j) \Big) P_r^{(\frac{d-2}{2})}(u), \quad u \in [-1,1], \tag{9.58}$$

where $a_j(n) = \frac{\Gamma(\frac{d-2}{2})\Gamma(n-2j+\frac{d-2}{2})\Gamma(j+\frac{1}{2})\Gamma(n-j+\frac{d-1}{2})}{\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2})j!\Gamma(n-j+\frac{d}{2})} \sim \frac{n-2j+1}{(n-j+1)^{1/2}(j+1)^{1/2}}$, for some constants depend only on d. The expansion (9.58) yields from the (8) in [4] after splitting between even and odd orders, changing the raw of summation and changing the variables. We skip the details. An expansion in a "closed" form is still under question. Let us add that the case of the interval $\mathbb{B}^1 = [-1,1]$, under general Jacobi-type weights $(1-u)^\alpha(1+u)^\beta$ is not covered in theory so far. Investigating this seems to be very interesting.

The most obvious direction of interest, which has been spotted also by Leonenko, Malyarenko and Olenko [26], is the study of spatio-temporal random fields on the ball. This would demand a comprehensive work for random fields on $\mathbb{B}^d \times \mathbb{R}$. More product domains, such as $\mathbb{B}^{d_1} \times \mathbb{R}^{d_2}$, $\mathbb{B}^{d_1} \times \mathbb{S}^{d_2}$ and $\mathbb{B}^{d_1} \times \mathbb{B}^{d_2}$, could be important to be explored too.

A grand challenge in multivariate random fields on the sphere, the ball or other manifolds, is the correspondence between the regularity of the covariance and the random field itself as Kerkyacharian, Ogawa, Petrushev and Picard did for the univariate case in [20]. This is something that would demand significant background to be built first, like regularity of vector-valued functions on manifolds and the necessary technology in the area.

Finally, the master problem that the community aims to face, is the meaningful relaxation of the assumption of isotropy. Of course such an assumption is pretty acceptable in the CMB framework, but may fail away from it.

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10. Appendix

In this section we have delivered all the proofs of our results, together with some details about simulations' studies.

Proof of the results in Section 3. We start with the regularity results.

Proof. (Of Theorem 3.1). Let $\lambda \geq 0$ and $0 \leq \nu \leq s$. Then

$$
\begin{aligned}
\left\langle \mathbf{C}^{(\nu)}, \mathbf{C}^{(\nu)} \right\rangle_{L^2_{(\lambda+\nu;m)}} &= \sum_{n,n'\geq 0} \left\langle \mathbf{B}_n \big(P_n^{(\lambda)}\big)^{(\nu)}, \mathbf{B}_{n'} \big(P_{n'}^{(\lambda)}\big)^{(\nu)} \right\rangle_{L^2_{(\lambda+\nu;m)}} \\
&= \sum_{n,n'\geq \nu} \eta(\lambda,n,\nu)\eta(\lambda,n',\nu) \left\langle \mathbf{B}_n P_{n-\nu}^{(\lambda+\nu)}, \mathbf{B}_{n'} P_{n'-\nu}^{(\lambda+\nu)} \right\rangle_{L^2_{(\lambda+\nu;m)}} \\
&= \sum_{n,n'\geq \nu} \eta(\lambda,n,\nu)\eta(\lambda,n',\nu) \operatorname{trace}\big(\mathbf{B}_n \mathbf{B}'_{n'}\big) \times \\
&\qquad \times \int_{-1}^{1} P_{n-\nu}^{(\lambda+\nu)}(u) P_{n'-\nu}^{(\lambda+\nu)}(u) (1-u^2)^{\lambda+\nu-\frac{1}{2}} du \\
&\sim \sum_{n=\nu}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2\nu+\mathfrak{t}(\lambda)} \qquad (10.59)
\end{aligned}
$$

where above we used the expansion (2.20), the properties (2.10), (2.8) and (2.8) of Gegenbauer/Chebyshev polynomials and the definition (3.23) of the Frobenius norm.

Now we replace the above in (3.25) and derive that

$$
\begin{aligned}
\|\mathbf{C}\|^2_{W^s_{(\lambda;m)}} &= \sum_{\nu=0}^{s} \|\mathbf{C}^{(\nu)}\|^2_{L^2_{(\lambda+\nu;m)}} \leq c \sum_{\nu=0}^{s} \sum_{n=\nu}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2\nu+\mathfrak{t}(\lambda)} \\
&\leq c(s+1) \sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2s+\mathfrak{t}(\lambda)} < \infty.
\end{aligned}
$$

Conversely, by (3.25) and (10.59), we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2s+\mathfrak{t}(\lambda)} \\
&\quad = \sum_{n<s} \|\mathbf{B}_n\|_F^2 (n+1)^{2s+\mathfrak{t}(\lambda)} + \sum_{n=s}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2s+\mathfrak{t}(\lambda)} \\
&\quad \leq s^{2s} \sum_{n=0}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{\mathfrak{t}(\lambda)} + \sum_{n=s}^{\infty} \|\mathbf{B}_n\|_F^2 (n+1)^{2s+\mathfrak{t}(\lambda)} \\
&\quad \leq c \sum_{\nu=0}^{s} \|\mathbf{C}^{(\nu)}\|^2_{L^2_{(\lambda+\nu;m)}} = c\|\mathbf{C}\|^2_{W^s_{(\lambda;m)}},
\end{aligned}
$$

which completes the proof.

□

Proof. (Of Corollaries 3.2 and 3.3). We just apply Theorem 3.1 for $\lambda = \frac{d-2}{2}$ and $\lambda = \frac{d-1}{2}$ respectively and noting that

$$
\mathfrak{t}\Big(\frac{d-2}{2}\Big) = \begin{cases} d-4, & d \geq 3 \\ 0, & ,d = 2 \end{cases} . \qquad (10.60)
$$

□

Some information about simulations in Section 4. In Section 4 we presented simulations experiments for visualizing multivariate isotropic random fields on the ball. We are placing here some more details.

For the case of the unit disk $\mathbb{B}^2$, we use a grid of size 500×500 for the polar coordinates' domain rescaled on $[0,1]^2$. Then we truncate to $N=100$ terms the series in (2.19). The orthogonal polynomials needed are available via the R-package "orthopolynom". For more see also the package "Rcosmos" [17]. Similarly for the 3D-unit ball $\mathbb{B}^3$ we used a grid of side $k=200$ for the spherical coordinates' domain adapted to present only the first octant of the ball. Then we rotated the plot for better implementation.

Let us put some more light on the bivariate random fields; $m=2$. This time we need a sequence $(\mathbf{B}_n)$ of positive definite 2×2 matrices. Let (b_n) a sequence of positive numbers and $-1<r<1$. We define

$$\mathbf{B}_n = b_n \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}. \tag{10.61}$$

Then $\mathbf{B}_n$ is positive definite, $\|\mathbf{B}_n\|_F = \sqrt{2(1+r^2)}b_n$ and $r=\mathrm{cor}(Z_1,Z_2)$, where $(Z_1,Z_2)'=\mathbf{Z}$ the random field as in (2.19). Moreover the square root of $\mathbf{B}_n$ is as follows

$$\mathbf{B}_n^{\frac{1}{2}} = \sqrt{b_n}\begin{pmatrix} 1 & 0 \\ r & \sqrt{1-r^2} \end{pmatrix}. \tag{10.62}$$

By entering the above square root in R directly, we "buy" some computational time, which is a vital issue for allowing us to increase the grid size. Otherwise R can find the square root with Cholesky decomposition too. The uniformly distributed $\mathbf{U}_n$ on the sphere $\mathbb{S}^d$ are extracted in the classical way: we simulate a $(d+1)$-dimensional standard normal and we normalize it by dividing with its vector norm.

Here we employed the sequence $b_n=(n+1)^{-\beta}$ for several β's interplaying with the regularity level and several values of r for expressing the correlation.

Proofs of the results in Section 5. We proceed to the Hölder continuity results.

Proof. (Of Theorem 5.1). We denote by $\lambda=\frac{d-2}{2}$. By (2.20) we have the following:

$$\begin{aligned} \|\mathbf{C}(u)\|_F &\le \sum_{n=0}^{\infty}\|\mathbf{B}_n\|_F|P_n^{(\lambda)}(u)| \le \sum_{n=0}^{\infty}\|\mathbf{B}_n\|_F|P_n^{(\lambda)}(1)| \\ &\le c\sum_{n=0}^{\infty}\|\mathbf{B}_n\|_F(n+1)^{(d-3)_+} \le c\sum_{n=0}^{\infty}\|\mathbf{B}_n\|_F(n+1)^{2\delta+(d-3)_+}<\infty, \end{aligned} \tag{10.63}$$

for every $u\in[-1,1]$; in the light of (2.9) and (5.38).

We proceed to the second term in (5.37). Let $-1\le u\ne t\le 1$. By (2.20) it turns out that

$$\|\mathbf{C}(u)-\mathbf{C}(t)\|_F \le \sum_{n=1}^{\infty}\|\mathbf{B}_n\|_F|P_n^{(\lambda)}(u)-P_n^{(\lambda)}(t)|. \tag{10.64}$$

We will estimate $|P_n^{(\lambda)}(u)-P_n^{(\lambda)}(t)|$ in two ways and interpolate the outcomes.

By mean value Theorem, there is a ξ between u and t such that

$$\begin{aligned} |P_n^{(\lambda)}(u)-P_n^{(\lambda)}(t)| &= \left|\left(\frac{d}{dr}P_n^{(\lambda)}\right)(\xi)\right||u-t| \le \max_{r\in[-1,1]}\left|\frac{d}{dr}P_n^{(\lambda)}(r)\right||u-t| \\ &= \eta(\lambda,n,1)\max_{r\in[-1,1]}|P_{n-1}^{(\lambda+1)}(r)||u-t| \\ &\le c(n+1)^{(d-3)_++2}|u-t|, \end{aligned} \tag{10.65}$$

in the light of the properties (2.10), (2.9) and (2.11) of Gegenbauer polynomials.

On the other hand we can estimate the same quantity as follows

$$\big|P_n^{(\lambda)}(u) - P_n^{(\lambda)}(t)\big| \le 2 \max_{u\in[-1,1]} \big|P_n^{(\lambda)}(u)\big| \le c(n+1)^{(d-3)_+}. \tag{10.66}$$

We interpolate (10.65) and (10.66) for the given $0 < \delta \le 1$ and we arrive at

$$\big|P_n^{(\lambda)}(u) - P_n^{(\lambda)}(t)\big| \le c(n+1)^{(d-3)_+ + 2\delta}|u-t|^{\delta}. \tag{10.67}$$

The combination of (10.64) and (10.67) together with (10.63) and the assumption (5.38) implies that

$$\big\|\mathbf{C}\big\|_{\mathcal{H}_m^\delta} \le c\sum_{n=0}^{\infty} \big\|\mathbf{B}_n\big\|_F (n+1)^{2\delta+(d-3)_+} < \infty.$$

□

Proof. (Of Corollary 5.2).

(a) Let $0 < \delta < \frac{1}{2}\big(s - \frac{d-1}{2}\big)$ and

$$\zeta := s + \frac{1}{2}\mathfrak{t}\Big(\frac{d-2}{2}\Big) - (d-3)_+ - 2\delta = s - 2\delta - \frac{d-2}{2} > \frac{1}{2}.$$

Then by the Cauchy-Schwarz inequality we derive

$$\begin{aligned}\big\|\mathbf{C}\big\|_{\mathcal{H}_m^\delta} &\le c\sum_{n=0}^{\infty} \big\|\mathbf{B}_n\big\|_F (n+1)^{2\delta+(d-3)_+}\\ &\le c\Big(\sum_{n=0}^{\infty}\frac{1}{(n+1)^{2\zeta}}\Big)^{\frac{1}{2}}\Big(\sum_{n=0}^{\infty}\big\|\mathbf{B}_n\big\|_F^2 (n+1)^{2s+\mathfrak{t}(\frac{d-2}{2})}\Big)^{\frac{1}{2}}\\ &\le c_{s,\delta}\big\|\mathbf{C}\big\|_{W^s_{(\frac{d-2}{2};m)}} < \infty.\end{aligned}$$

(b) The claim follows by Corollary 3.2 and the standard inclusion $\ell^2 \hookrightarrow \ell^1$ and exactly because $s + \mathfrak{t}(\frac{d-2}{2}) \le 2\delta + (d-3)_+$;

$$\begin{aligned}\big\|\mathbf{C}\big\|_{W^s_{(\frac{d-2}{2};m)}} &\le c\sum_{n=0}^{\infty}\big\|\mathbf{B}_n\big\|_F (n+1)^{s+\frac{1}{2}\mathfrak{t}(\frac{d-2}{2})}\\ &\le c\sum_{n=0}^{\infty}\big\|\mathbf{B}_n\big\|_F (n+1)^{2\delta+(d-3)_+} < \infty.\end{aligned}$$

□

Proofs of the results in Section 6. We are ready to proof the Theorems stated in Section 6.

Proof. (Of Theorem 6.1). Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d$. We denote by $\mathbf{W}_{12} = \mathbf{Z}(\mathbf{x}_1) - \mathbf{Z}(\mathbf{x}_2)$. Then $\mathbb{E}(\mathbf{W}_{12}) = \mathbf{0}$ and

$$\begin{aligned}\operatorname{cov}\big(\mathbf{W}_{12}\big) &= \mathbb{E}\big(\mathbf{W}_{12}\mathbf{W}_{12}'\big)\\ &= \mathbb{E}\big(\mathbf{Z}(\mathbf{x}_1)\mathbf{Z}(\mathbf{x}_1)'\big) - \mathbb{E}\big(\mathbf{Z}(\mathbf{x}_1)\mathbf{Z}(\mathbf{x}_2)'\big)\\ &\quad - \mathbb{E}\big(\mathbf{Z}(\mathbf{x}_2)\mathbf{Z}(\mathbf{x}_1)'\big) + \mathbb{E}\big(\mathbf{Z}(\mathbf{x}_2)\mathbf{Z}(\mathbf{x}_2)'\big)\\ &= 2\big(\mathbf{C}(1) - \mathbf{C}\big(\cos(\rho(\mathbf{x}_1,\mathbf{x}_2))\big)\big) =: \mathbf{\Sigma}_0,\end{aligned} \tag{10.68}$$

thanks to the isotropy assumption (2.4).

Then the random vector $\mathbf{Y} = \mathbf{\Sigma}_0^{-\frac{1}{2}}\mathbf{W}_{12}$ is standard m-variate Gaussian; $\mathbf{Y} \sim \mathcal{N}_m(\mathbf{0}, \mathbf{I}_m)$ and by $\mathbf{\Sigma}_0^{-\frac{1}{2}}$ we denoted the inverse of the square root of the positive definite matrix $\mathbf{\Sigma}_0$.

By Cauchy-Schwarz inequality we can see that

$$\left\|\mathbf{W}_{12}\right\|_F^{2p} = \left(\left\|\boldsymbol{\Sigma}_0^{\frac{1}{2}}\mathbf{Y}\right\|_F^2\right)^p \leq \left\|\boldsymbol{\Sigma}_0^{\frac{1}{2}}\right\|_F^{2p}\left\|\mathbf{Y}\right\|_F^{2p}. \tag{10.69}$$

We focus first on the term $\left\|\boldsymbol{\Sigma}_0^{\frac{1}{2}}\right\|_F^{2p}$.

By (3.23), the fact that $\boldsymbol{\Sigma}_0^{\frac{1}{2}}$ is the square root of the positive definite matrix $\boldsymbol{\Sigma}_0$ and (10.68), we obtain

$$\begin{aligned}\left\|\boldsymbol{\Sigma}_0^{\frac{1}{2}}\right\|_F^2 &= \operatorname{trace}\Big(\boldsymbol{\Sigma}_0^{\frac{1}{2}}\big(\boldsymbol{\Sigma}_0^{\frac{1}{2}}\big)'\Big)\\ &= \operatorname{trace}\big(\boldsymbol{\Sigma}_0\big) = 2\operatorname{trace}\Big(\mathbf{C}(1) - \mathbf{C}\big(\cos(\rho(\mathbf{x}_1,\mathbf{x}_2))\big)\Big).\end{aligned} \tag{10.70}$$

Now given that $\mathbf{C}$ enjoys (5.38), we apply Theorem 6.1 and we derive

$$\begin{aligned}\operatorname{trace}\Big(\mathbf{C}(1) - \mathbf{C}\big(\cos(\rho(\mathbf{x}_1,\mathbf{x}_2))\big)\Big) &\leq c_m\left\|\mathbf{C}(1) - \mathbf{C}\big(\cos(\rho(\mathbf{x}_1,\mathbf{x}_2))\big)\right\|_F\\ &\leq c\left\|\mathbf{C}\right\|_{\mathcal{H}_m^\delta}\left|1-\cos(\rho(\mathbf{x}_1,\mathbf{x}_2))\right|^\delta\\ &\leq c\left\|\mathbf{C}\right\|_{\mathcal{H}_m^\delta}\rho(\mathbf{x}_1,\mathbf{x}_2)^{2\delta},\end{aligned} \tag{10.71}$$

since $|1-\cos(t)| \leq ct^2$, for $t \in [-1,1]$.

On the other hand we observe that $\|\mathbf{Y}\|_F^2 = \sum_{i=1}^m Y_i^2$. Then

$$\|\mathbf{Y}\|_F^{2p} = \left(\sum_{i=1}^m Y_i^2\right)^p = \sum_{k_1+\cdots+k_m=p}\binom{p}{k_1,\ldots,k_m}Y_1^{2k_1}\cdots Y_m^{2k_m}.$$

By the fact that $\mathbf{Y}$ is standard Gaussian, the Y_i are independent to each other, therefore we extract that

$$\begin{aligned}\mathbb{E}\big(\|\mathbf{Y}\|_F^{2p}\big) &= \sum_{k_1+\cdots+k_m=p}\binom{p}{k_1,\ldots,k_m}\mathbb{E}\big(Y_1^{2k_1}\big)\cdots\mathbb{E}\big(Y_m^{2k_m}\big)\\ &= \sum_{k_1+\cdots+k_m=p}\binom{p}{k_1,\ldots,k_m}\prod_{i=1}^m(2k_i-1)!! =: c_{m,p},\end{aligned} \tag{10.72}$$

where we used the well known property for (univariate) standard normal distribution Y that the even order moments satisfy

$$\mathbb{E}(Y^{2k}) = (2k-1)!! = 1\cdot 3\cdots(2k-1). \tag{10.73}$$

Now (6.41) is a consequence of (10.69)-(10.72).

□

Proof. (Of Theorem 6.3). We will employ Theorem 1 from [34].

Let $\gamma < \delta$. Let $p \in \mathbb{N}$ be such that the integer $q = 2p$ to satisfy $\frac{d}{q} < \delta - \gamma$. We denote by $\beta = \delta q - d$. Then $\beta > 0$ and $\gamma < \frac{\beta}{q}$.

By Theorem 6.1 for the integer p we have

$$\mathbb{E}\Big(\left\|\mathbf{Z}(\mathbf{x}_1) - \mathbf{Z}(\mathbf{x}_2)\right\|_F^q\Big) \leq c\rho(\mathbf{x}_1,\mathbf{x}_2)^{\delta q} = c\rho(\mathbf{x}_1,\mathbf{x}_2)^{d+\beta}.$$

The last implies that there exists a modification $\big\{\tilde{\mathbf{Z}}(\mathbf{x}) : \mathbf{x} \in \mathbb{B}^d\big\}$ of $\mathbf{Z}$, which is locally Hölder continuous of order γ, because of the equivalence of the ball distance with the standard Euclidean distance on $\mathbb{R}^d$ (which can been derived as in [24]). □

Proof of the results in Section 7. We can now present the proof of the approximation Theorem 7.1:

Proof. In the proof we focus on the expansion (2.16) and the corresponding truncation (7.44). The handling for the expansion (2.19) is completely similar.

(a) We start by proving (7.50) for $p = 2$. In this case the error on the left can be simply expressed as in (7.48). Let $N \geq n_0$. We denote for brevity $\ell = \frac{d-1}{2}$ and $\mathbf{D}_n = \mathbf{B}_n^{1/2}$. By (2.16) and (7.44) we derive

$$\big\langle \mathbf{Z}(\mathbf{x}) - \mathbf{Z}_N(\mathbf{x}) \big\rangle_F = \sum_{n,k>N} \operatorname{trace}\big(\mathbf{D}_n \mathbf{V}_n \mathbf{V}_k' \mathbf{D}_k'\big) P_n^{(\ell)}\big(\mathbf{y}'\mathbf{U}\big) P_k^{(\ell)}\big(\mathbf{y}'\mathbf{U}\big), \quad \mathbf{y} = \mathbf{y}(\mathbf{x}).$$

This implies that

$$\mathbb{E}\Big(\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_F^2\Big) = \sum_{n,k>N} \operatorname{trace}\big(\mathbf{D}_n \mathbb{E}\big(\mathbf{V}_n \mathbf{V}_k'\big) \mathbf{D}_k'\big) \times \mathbb{E}\Big(P_n^{(\ell)}\big(\mathbf{y}'\mathbf{U}\big) P_k^{(\ell)}\big(\mathbf{y}'\mathbf{U}\big)\Big), \tag{10.74}$$

by the independence of $\mathbf{V}_n$, $\mathbf{U}$.

Since $\{\mathbf{V}_n\}$ are independent and have zero mean and covariance given in (2.14), we extract that

$$\begin{aligned} \operatorname{trace}\big(\mathbf{D}_n \mathbb{E}\big(\mathbf{V}_n \mathbf{V}_k'\big) \mathbf{D}_k'\big) &= \delta_{nk} \operatorname{trace}\big(\mathbf{D}_n \mathbb{E}\big(\mathbf{V}_n \mathbf{V}_n'\big) \mathbf{D}_n'\big) \\ &= \delta_{nk} a_n^2 \operatorname{trace}\big(\mathbf{D}_n \mathbf{D}_n'\big) \\ &= \delta_{nk} a_n^2 \operatorname{trace}\big(\mathbf{B}_n\big) \end{aligned} \tag{10.75}$$

Since $\mathbf{U}$ is uniformly distributed on $\mathbb{S}^d$ and $\mathbf{y} \in \mathbb{S}^d$, the identity (2.12) implies

$$\begin{aligned} \mathbb{E}\Big(P_n^{(\ell)}\big(\mathbf{y}'\mathbf{U}\big) P_n^{(\ell)}\big(\mathbf{y}'\mathbf{U}\big)\Big) &= \frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} P_n^{(\ell)}\big(\mathbf{y}'\mathbf{u}\big) P_n^{(\ell)}\big(\mathbf{y}'\mathbf{u}\big) d\mathbf{u} \\ &= a_n^{-2} P_n^{(\ell)}(1) \sim a_n^{-2} n^{d-2}, \end{aligned} \tag{10.76}$$

thanks to the behaviour (2.9).

Combining (10.74)-(10.76) we have that the error has the form

$$\big\|\mathbf{Z} - \mathbf{Z}_N\big\|^2_{L^2(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))} \sim \mu(\mathbb{B}^d) \sum_{n>N} \operatorname{trace}\big(\mathbf{B}_n\big) n^{d-2}. \tag{10.77}$$

Replacing assumption (7.49) to (10.77) we arrive at

$$\big\|\mathbf{Z} - \mathbf{Z}_N\big\|^2_{L^2(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))} \leq c \sum_{n>N} n^{-1-\varepsilon} \leq c \int_N^\infty \frac{dx}{x^{1+\varepsilon}} \leq c N^{-\varepsilon}, \tag{10.78}$$

which completes the proof of (a) for $p = 2$.

Let $p < 2$, therefore $2/p > 1$. By Hölder's inequality for this index on the probability measure and (10.78) we arrive at

$$\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^p(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))} \leq \big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^2(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))} \leq c N^{-\varepsilon/2}.$$

Let $p > 2$ and let $\nu = \nu(p)$ be the (unique) integer such that $2(\nu - 1) < p \leq 2\nu$. By Hölder's inequality for the index $2\nu/p$ we obtain

$$\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^p(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))} \leq \big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^{2\nu}(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))}. \tag{10.79}$$

We apply Corollary 2.17 of [15] deriving the existence of a constant $c_\nu > 0$ such that

$$\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^{2\nu}(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))} \leq c_\nu \big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^2(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))}. \tag{10.80}$$

Now we use (10.79), (10.80) and (10.75) to conclude that for every $p > 2$

$$\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^p(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))} \le c\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^2(\Omega, L^2(\mathbb{B}^d;\mathbb{R}^m))} \le cN^{-\varepsilon/2},$$

which completes the proof of claim (a).

(b) Let $0 < \alpha < \varepsilon/2$. We need to prove that

$$\mathbb{P}\big(\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^2(\mathbb{B}^d)} \ge N^{-\alpha}, \text{ for infinity many } N \in \mathbb{N}\big) = 0. \tag{10.81}$$

We apply the Borel-Cantelli's lemma [1, Theorem 2.7]. It suffices to prove that

$$\sum_{N=n_0}^{\infty} \mathbb{P}\big(\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^2(\mathbb{B}^d)} \ge N^{-\alpha}\big) < \infty. \tag{10.82}$$

Let $p > 0$. By Chebyshev's inequality, (7.50) and using the mixed-norm (7.47), we derive

$$\begin{aligned}\mathbb{P}\big(\big\|\mathbf{Z} - \mathbf{Z}_N\big\|_{L^2(\mathbb{B}^d)} \ge N^{-\alpha}\big) &\le N^{\alpha p}\mathbb{E}\big(\big\|\mathbf{Z} - \mathbf{Z}_N\big\|^p_{L^2(\mathbb{B}^d)}\big)\\ &= N^{\alpha p}\big\|\mathbf{Z} - \mathbf{Z}_N\big\|^p_{L^p(\Omega, L^2(\mathbb{B}^d))}\\ &\le cN^{-(-\alpha+\varepsilon/2)p}. \end{aligned}\tag{10.83}$$

We choose now $p > 1/(-\alpha + \varepsilon/2)$ and we obtain

$$\sum_{N=1}^{\infty} N^{-(-\alpha+\varepsilon/2)p} < \infty.$$

The last together with (10.83) leads to (10.82) and completes the proof. □

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