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STOCHASTIC PROCESS GENERATED BY 1-D ISING MODEL WITH COMPETING INTERACTIONS

We consider a stochastic process generated by 1-D Ising model with competing interactions and describe all distributions of this process. It is shown that the set of all limit Gibbs measures, i.e. phase diagram, consist of ferromagnetic, antiferromagnetic, paramagnetic and modulated phases. Also it is proven that on the set of ferromagnetic phases one can reach the phase transition.

1. Introduction

In the theory of random processes, every process is determined by the family of its finite-dimensional probability distributions. By Kolmogorov's fundamental theorem, these distributions give rise to a unique probability measure on the $\sigma-$algebra of measurable subsets generated by the finite-dimensional cylindrical sets.

In problems of classical statistical mechanics we consider a Gibbs measure that is a mathematical idealization of an equilibrium state of a physical system which consists of a very large number of interacting components. In the language of Probability Theory, a Gibbs measure is simply the distribution of a stochastic process which, instead of being indexed by the time, is parametrized by the sites of a spatial lattice, and has the special feature of admitting prescribed versions of the conditional distributions with respect to the configurations outside finite regions. Then the physical phenomenon of phase transition should be reflected by the non-uniqueness of the Gibbs measures for fixed configurations outside finite regions.

The Ising model represents a simplified mathematical description of interacting magnetic spins within a lattice structure. It assumes that each spin can take one of two possible values: up or down, representing the spin alignment of individual atoms or molecules. The interactions between neighbouring spins are considered, leading to the emergence of collective behaviour and the manifestation of macroscopic magnetic properties.

The first formulation of the Ising model given by Ising himself is as follows [6]: consider a sequence $\Lambda_n = \{0, 1, 2, \cdots, n\}$ of points on the line. At each point, or site, there is a small dipole or "spin" which at any given moment is in one of two positions, "up" or "down". It is indicated the spins in the form of a configuration $\omega = (\omega_0, \omega_1, \cdots, \omega_n)$, where $\omega_j = +$ or $-$ with "+" indicating a spin up and "-" a spin down and the spin σ_j is defined as a function $\sigma_j : \Omega_n \to \{-1, 1\}$ such that $\sigma_j(\omega) = 1$ if $\omega_j = +$ and $\sigma_j(\omega) = -1$ if $\omega_j = -$.

Let Ω_n be a space of all possible configurations $\omega = (\omega_0, \omega_1, \cdots, \omega_n)$ defined on the set Λ_n. Ising defined a probability measure on Ω_n as follows. To each configuration ω an energy $H(\omega)$ is assigned by

$$H(\omega) = -J \sum_{i,j:i<j} \sigma_i(\omega)\sigma_j(\omega) \tag{1}$$

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The sum represents the energy caused by interaction of the spins. Ising made the simplifying assumption that only interaction between neighbouring spins i and $i+1$ need be taken into account. Such pairs is called nearest neighbours and denoted as $< i, i+1 >$. Thus one can rewrite the equality (1) as following

$$H(\omega) = -J \sum_{i=0}^{n-1} \sigma_i(\omega)\sigma_{i+1}(\omega) \tag{2}$$

Ising then assigned probabilities to configurations ω proportional to

$$e^{-\frac{1}{kT}U(\omega)},$$

where T is the absolute temperature (Kelvin scale), k is a universal constant and $\beta = \frac{1}{kT}$ is a positive number which is proportional to the inverse of the absolute (Kelvin) temperature. The probability measure on Ω_n is thus given by

$$P(\omega) = \frac{e^{-\beta H(\omega)}}{Z},$$

where the normalizing constant Z, defined by

$$Z = \sum_{\omega} e^{-\beta H(\omega)}$$

is called the partition function.

Such probability measure defined by an energy function H is called a Gibbs measure. It is evident that for finite set Λ_n there exists unique Gibbs measure.

Now consider Ising model defined on infinite countable set $Z_+ = \{0, 1, 2, \cdots\}$. Assume $\omega : Z_+ \to \{-, +\}$ is a configuration on Z_+ and Ω is the set of all configurations defined on Z_+. In this case the collection of all random variables $\{\sigma_j\}_{j=0}^{\infty}$ forms a stochastic process (random field). Now we consider conditional probability distributions of the random process or field inside any finite domain under the condition that its values outside the domain are fixed.

Let $\Lambda_n = \{0, 1, \cdots, n\}$. It is evident $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \cdots$ and $\cup_{i=1}^{\infty}\Lambda_i = Z_+$.

As above we consider configurations $\omega : Z_+ \to \{+, -\}$. The restriction of ω on any subset $\Lambda_n \subset Z_+$ is denoted by $\omega_n(\Lambda_n)$.

Any configuration ω one can split into two sub-configurations $\omega_n(\Lambda_n)$ and $\omega^n(Z_+ \setminus \Lambda_n)$, where $Z_+ \setminus \Lambda_n$ is still infinite set.

Let us fix some configuration $\bar{\omega}^n(Z_+ \setminus \Lambda_n) : Z_+ \setminus \Lambda_n \to \{+, -\}$ and call it boundary configuration. Then we consider the set of all configurations that vary on Λ_n but fixed on $Z_+ \setminus \Lambda_n$. Note that this set is finite. The basic assumption concerning the models that we will consider is the following: if we know what is happening outside a finite subset Λ_n of then we can compute the distribution of configurations on the finite set. We define the conditional Hamiltonian with fixed boundary configuration $\bar{\omega}^n(Z_+ \setminus \Lambda_n)$ as follows

$$H(\omega_n|\bar{\omega}^n) = -J \sum_{i=0}^{n-1} \sigma_i(\omega_n)\sigma_{i+1}(\omega_n) - J\sigma_n(\omega_n)\sigma_{n+1}(\bar{\omega}^n) \tag{3}$$

Then the conditional Gibbs state on finite subset Λ_n with boundary configuration $\bar{\omega}^n(Z_+ \setminus \Lambda_n)$ to be the measure μ_n given by

$$\mu_n(\omega_n(\Lambda_n)|\bar{\omega}^n(Z_+ \setminus \Lambda_n)) = \frac{e^{-\beta H(\omega_n|\bar{\omega}^n)}}{Z_n(\bar{\omega}^n)} \tag{4}$$

for any configuration $\omega_n(\Lambda_n) \in \Omega_n(\Lambda_n)$, where

$$Z_n(\bar{\omega}^n) = \sum_{\omega_n(\Lambda_n) \in \Omega_n(\Lambda_n)} e^{-\beta H(\omega_n(\Lambda_n)|\bar{\omega}^n(Z_+ \setminus \Lambda_n))}.$$

Now we will define a limit Gibbs state on Ω by the following way.

We will say that μ is a limit Gibbs state on Ω, if for any finite subset Λ_n and for arbitrary boundary configuration $\bar{\omega}^n(Z_+ \setminus \Lambda_n)$ the conditional probability with respect to μ given that the configuration $\omega \in \Omega$ is $\bar{\omega}^n(Z_+ \setminus \Lambda_n)$ on $Z_+ \setminus \Lambda_n$ is the same as the conditional Gibbs state on $\Omega(Z_+)$ given above:

$$\mu(\omega(\Lambda_n)|\bar{\omega}^n(Z_+ \setminus \Lambda_n)) = \frac{e^{-\beta H(\omega_n|\bar{\omega}^n)}}{Z_n(\bar{\omega}^n)} \tag{5}$$

The main problem of equilibrium statistical physics is to describe all limit Gibbs states of given Hamiltonian, i.e. to investigate the existence and uniqueness of such measures. The phenomenon of non-uniqueness of a Gibbs measure can be interpreted as a phase transition and is, as such, of particular physical significance.

Ising [6] proved that for considered 1-D Hamiltonian there exists unique limit Gibbs distribution, that is this limit Gibbs measure does not depend from choice of boundary configuration $\bar{\omega}^n$.

1.1. **Nonhomogeneous 1-D Ising model.** 1-D inhomogeneous Ising model is defined by the following Hamiltonian

$$H(\omega) = -\sum_{n=1}^{\infty} J_n \sigma_n(\omega)\sigma_{n+1}(\omega) \tag{6}$$

In [2] it is proven that if $J_n > 0$ for all $n \geq 1$ and

$$\sum_{n=1}^{\infty} e^{-2J_n} < \infty$$

then the measure μ_+ generated by positive boundary configurations $\bar{\omega}^n(k) \equiv +$ for all $k > n$ is not equal to the measure μ_- generated by negative boundary configurations $\bar{\omega}^n(k) \equiv -$ for all $k > n$.

For instance, one might take $J_n = clog(l+n)$ with $c > 1/2$.

2. 1-D Ising model with competing interactions

For Hamiltonian of considered above homogeneous Ising model

$$H(\omega) = -J\sum_{n=0}^{\infty} \sigma_n(\omega)\sigma_{n+1}(\omega) \tag{7}$$

we add one more interaction, namely interactions of second neighbours $> i, i+2 <$ with strength J_2

$$H(\omega) = -J_1\sum_{i=0}^{\infty} \sigma_i(\omega)\sigma_{i+1}(\omega) - J_2\sum_{i=0}^{\infty} \sigma_i(\omega)\sigma_{i+2}(\omega). \tag{8}$$

Such model is called the Ising model with competing interactions J_1 and J_2.

The existence of competing interactions lies at the heart of a variety of original phenomena in magnetic systems, ranging from the spin-glass transitions found in many disordered materials to the modulated phases with an infinite number of commensurate regions, that are observed in certain models with periodic interactions.

Axial Next-Nearest-Neighbour Ising model on Z^2 [1],[3],[4], much studied in connection with materials like CeSB, where interactions of different signs are in conflict along one direction.

A Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles with exactly $k+1$ edges issuing from each vertex and respectively a semi-infinite Cayley tree Γ^k_+ of order k is the infinite graph without cycles with $k+1$ edges issuing from each vertex except for x^0 which has only k edges. In this case the vertex x^0 is called a root of this tree.

An advantage of models on trees is that no approximations have to be made and the calculations can be carried out with high accuracy [5],[8],[9] . The important point is that statistical mechanics on trees involve non-linear recursion equations and are naturally connected to the rich world of dynamical systems, a world presently under intense investigation.

A graph on Z_+ one can consider as semi-infinite Cayley tree of first order, that is a connected graph without cycles where each vertex, except vertex $\{0\}$, has exactly 2 nearest neighbours, and vertex $\{0\}$ has only one nearest neighbour.

Let $\Lambda_n = \{0, 1, \cdots, n\}$, Ω_n is the set of all configurations and $\bar{\omega}^n(\Lambda_n^c)$ is a fixed boundary configuration.

Then a conditional Hamiltonian is defined as follows

$$\begin{aligned} H(\omega_n|\bar{\omega}^n) &= -J_1\sum_{i=0}^{n-1}\sigma_i(\omega_n)\sigma_{i+1}(\omega_n) - J_2\sum_{i=0}^{n-2}\sigma_i(\omega_n)\sigma_{i+2}(\omega_n) \\ &\quad -J_1\sigma_n(\omega_n)\sigma_{n+1}(\bar{\omega}^n) - J_2\sigma_{n-1}(\omega_n)\sigma_{n+1}(\bar{\omega}^n) \\ &\quad -J_2\sigma_n(\omega_n)\sigma_{n+2}(\bar{\omega}^n) \end{aligned}$$

Here last three terms interactions with boundary configuration $\bar{\omega}^n(\Lambda_n^c)$. Then the conditional Gibbs state on finite subset Λ_n with boundary configuration $\bar{\omega}^n(Z_+ \setminus \Lambda_n)$ to be the measure μ_n given by

$$\mu_n(\omega_n(\Lambda_n)|\bar{\omega}^n(Z_+ \setminus \Lambda_n)) = \frac{e^{-\beta H(\omega_n|\bar{\omega}^n)}}{Z_n(\bar{\omega}^n)} \tag{9}$$

for any configuration $\omega_n(\Lambda_n) \in \Omega_n(\Lambda_n)$, where

$$Z_n(\bar{\omega}^n) = \sum_{\omega_n(\Lambda_n)\in\Omega_n(\Lambda_n)} e^{-\beta H(\omega_n(\Lambda_n)|\bar{\omega}^n(Z_+\setminus\Lambda_n))}.$$

Let Ω_n is the set of all configurations on Λ_n. We split this set into four two-dimensional cylinder subsets $\Omega_n^{++}, \Omega_n^{+-}, \Omega_n^{-+}, \Omega_n^{--}$, where

$$\begin{aligned} \Omega_n^{++} &= \{\omega_n \in \Omega_n : \sigma_0(\omega_n) = +1, \sigma_1(\omega_n) = +1\} \\ \Omega_n^{+-} &= \{\omega_n \in \Omega_n : \sigma_0(\omega_n) = +1, \sigma_1(\omega_n) = -1\} \\ \Omega_n^{-+} &= \{\omega_n \in \Omega_n : \sigma_0(\omega_n) = -1, \sigma_1(\omega_n) = +1\} \\ \Omega_n^{--} &= \{\omega_n \in \Omega_n : \sigma_0(\omega_n) = -1, \sigma_1(\omega_n) = -1\} \end{aligned}$$

and respectively

$$\begin{aligned} Z_n^{++}(\bar{\omega}^n) &= \sum_{\omega_n\in\Omega_n^{++}} \exp(-\beta H(\omega_n|\bar{\omega}^n)) \\ Z_n^{+-}(\bar{\omega}^n) &= \sum_{\omega_n\in\Omega_n^{+-}} \exp(-\beta H(\omega_n|\bar{\omega}^n)) \\ Z_n^{-+}(\bar{\omega}^n) &= \sum_{\omega_n\in\Omega_n^{-+}} \exp(-\beta H(\omega_n|\bar{\omega}^n)) \\ Z_n^{--}(\bar{\omega}^n) &= \sum_{\omega_n\in\Omega_n^{--}} \exp(-\beta H(\omega_n|\bar{\omega}^n)) \end{aligned}$$

Assume for brevity

$$z_1 = Z_n^{++}(\bar{\omega}^n), z_2 = Z_n^{+-}(\bar{\omega}^n), z_3 = Z_n^{-+}(\bar{\omega}^n), z_4 = Z_n^{--}(\bar{\omega}^n)$$

and

$$z_1' = Z_{n+1}^{++}(\bar{\omega}^{n+1}), z_2' = Z_n^{+-}(\bar{\omega}^n), z_3' = Z_n^{-+}(\bar{\omega}^n), z_4' = Z_n^{--}(\bar{\omega}^n),$$

where $\bar{\omega}^{n+1} = \bar{\omega}^n|_{Z_+ \setminus \Lambda_{n+1}}$.

Let $a = \exp(2\beta J_1)$ and $b = \exp(2\beta J_2)$, with $\beta = \frac{1}{kT}$, where k is a universal constant and T is the temperature. Then one can produce the following recursion equations

$$\begin{aligned} z_1' &= abz_1 + ab^{-1}z_2 \\ z_2' &= a^{-1}bz_3 + a^{-1}b^{-1}z_4 \\ z_3' &= a^{-1}b^{-1}z_1 + a^{-1}bz_2 \\ z_4' &= ab^{-1}z_3 + abz_4 \end{aligned}$$

In the high temperature ($a \sim 1, b \sim 1$) we have paramagnetic phase, i.e. the spins are as likely to point up as down, whatever the initial conditions: one has $z_1 = z_4$ and $z_2 = z_3$, so that a possible choice of reduced (renormalized) variables is

$$\begin{aligned} x &= \frac{z_2 + z_3}{z_1 + z_4} \\ y_1 &= \frac{z_1 - z_4}{z_1 + z_4} \\ y_2 &= \frac{z_2 - z_3}{z_1 + z_4} \end{aligned}$$

According recursion equations for $z_i, i = 1, 2, 3, 4$ the previous relations one can rewrite as follows:

$$x' = \frac{1 + bx}{a(x + b)} \tag{10}$$

$$y_1' = \frac{by_1 + y_2}{x + b} \tag{11}$$

$$y_2' = -\frac{y_1 + by_2}{a(x + b)} \tag{12}$$

The average magnetization m for the nth generation is given by expectation $\sigma_0(\omega_n)$ on $\Omega(\Lambda_n)$ with respect to conditional Gibbs measure μ_n, that is

$$m = \frac{y_1 + y_2}{x}. \tag{13}$$

Starting from random initial conditions with $y_1, y_2 \neq 0$, one iterates the recursion relations (10-12) and observes their behaviour after a large number of iterations.

In the simplest situation a fixed point (x^*, y_1^*, y_2^*) is reached. According (13) it corresponds to a paramagnetic phase if $y_1^* = y_2^* = 0$, or to a ferromagnetic phase if $y_1^*, y_2^* \neq 0$. A limit cycle signals a commensurate state, with period a multiple of the distance between sites. Finally, the system may remain aperiodic, which corresponds to an incommensurate phase. The distinction between a truly aperiodic case and one with a very long period is difficult to make numerically.

Plotting a phase diagram on the plane $\gamma = -J_2/J_1, \alpha = T/(k_B J_1)$ one can show that the phase diagram contains a modulated phase, as found for similar models. The modulated phase contains both incommensurate regions and commensurate regions.

These results differ from those obtained in for Ising model with competing nearest-neighbour and second neighbour interactions considered on a Cayley tree of second order [9].

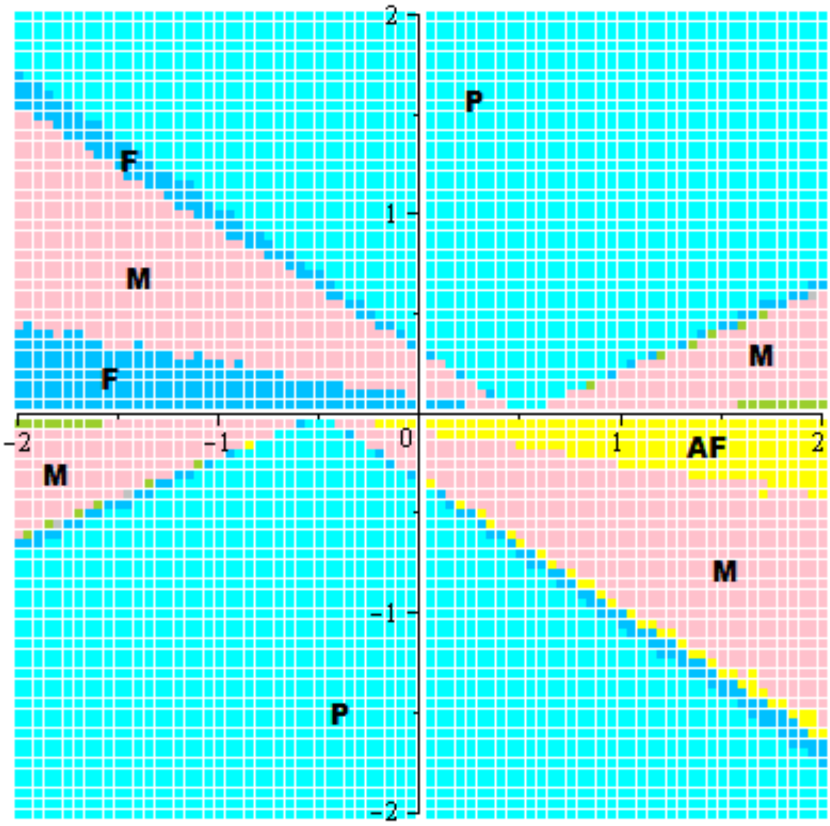

FIGURE 1. Phase diagram of the 1-D Ising model with competing interactions

3. Phase Transition

As noted above for 1-D Ising model we don't reach phase transition. Below we show for 1-D Ising model with competing interactions we reach phase transition on the set of ferromagnetic Gibbs states. It is easy to see that

$$\mu_n(\{\omega_n \in \Omega_n : \sigma_0(\omega_n) = +1\}|\bar{\omega}^n(Z_+ \setminus \Lambda_n)) = \frac{1 + x + y_1 + y_2}{2(1 + x)} \tag{14}$$

and respectively

$$\mu_n(\{\omega_n \in \Omega_n : \sigma_0(\omega_n) = -1\}|\bar{\omega}^n(Z_+ \setminus \Lambda_n)) = \frac{1 + x - y_1 - y_2}{2(1 + x)} \tag{15}$$

Then for limit Gibbs measure μ we have

$$\mu(\{\omega \in \Omega : \sigma_0(\omega) = +1\}) = \frac{1 + x^* + y_1^* + y_2^*}{2(1 + x^*)} \tag{16}$$

and respectively

$$\mu(\{\omega \in \Omega : \sigma_0(\omega) = -1\}) = \frac{1 + x^* - y_1^* - y_2^*}{2(1 + x^*)} \tag{17}$$

Since we are considering ferromagnetic phase, the fixed point (x^*, y_1^*, y_2^*) there exist with $y_1* \neq 0$ and $y_2* \neq 0$.

From Equations (11) and (12) we have

$$\frac{y_1'}{y_2'} = -\frac{a(by_1 + y_2)}{y_1 + by_2}$$

Let $\frac{y_1}{y_2} = t$. Then

$$t' = -\frac{a(bt+1)}{t+b}$$

The fixed points t^* one can find solving the following quadratic equation

$$t^2 + (1+a)bt + a = 0.$$

There exist two real roots if

$$b > \frac{2\sqrt{a}}{a+1}$$

With respect to the variables $\gamma = -J_2/J_1, \alpha = T/(k_B J_1)$ this inequality one can rewrite as

$$2\gamma < \alpha \ln \frac{e^{2\alpha^{-1}} + 1}{2e^{\alpha^{-1}}} \tag{18}$$

It is easy to verify that the solution of this inequality contains the domain of ferromagnetic phase. One can verify numerically that the measure μ_+ generated by positive boundary configurations $\bar{\omega}^n(k) \equiv +$ for all $k > n$ is not equal to the measure μ_- generated by negative boundary configurations $\bar{\omega}^n(k) \equiv -$ for all $k > n$.

Thus we have proved the following statement.

Theorem. *For the Ising model with competing interactions there exists a phase transition.*

As corollary one can reproof Ising result about non-existence phase transition for 1-D Ising model. Note that for Ising model with $J_2 = 0$, we have $b = 1$ and equations

$$y_1' = \frac{by_1 + y_2}{x+b}$$
$$y_2' = -\frac{y_1 + by_2}{a(x+b)}$$

one can rewrite as

$$y_1' = \frac{y_1 + y_2}{x+1}$$
$$y_2' = -\frac{y_1 + y_2}{a(x+1)}$$

that is

$$\frac{y_1'}{y_2'} = -a \tag{19}$$

Therefore there does not exist a phase transition.

Conclusion. It is shown that for 1-D Ising model with competing interactions with $J_2 \neq 0$ on the domain of ferromagnetic phases one can reach phase transition.

References

1. R.J. Elliott, *Phenomenological Discussion of Magnetic Ordering in the Heavy Rare-Earth Metals*, Phys.Rev. **124** (1961), 346–355. https://doi.org/10.1103/PhysRev.124.346
2. H.O. Georgii, *Gibbs Measures and Phase Transitions*, de Gruyter Studies in Mathematics, 9, Walter de Gruyter, Berlin, 2011. https://doi.org/10.1515/9783110250329
3. D.A. Huse, *Simple three-state model with infinitely many phases*, Phys. Rev. B **24** (1981), 5180–5194. https://doi.org/10.1103/PhysRevB.24.5180

4. S. Ostlund, *Incommensurate and commensurate phases in asymmetric clock models*, Phys. Rev. B **24** (1981), 398–405. https://doi.org/10.1103/PhysRevB.24.398
5. S. Inawashiro, C.J. Thompson, *Competing ising interactions and chaotic glass-like behaviour on a Cayley tree*, Phys. Lett. A **97** (1983), 245–248. https://doi.org/10.1016/0375-9601(83)90758-2
6. E. Ising, *Beitrag zur Theorie des Ferromagnetismus*, Z. Phys. **31** (1925), 253–258. https://doi.org/10.1007/BF02980577
7. R. Kindermann and J.L. Snell, *Markov random fields and their applications*, Vol.1, Providence, RI: American Mathematical Society, 1980. https://doi.org/10.1090/conm/001
8. A.M. Mariz, C. Tsallis, E.L. Albuquerque, *Phase diagram of an Ising model on a Cayley tree in the presence of competing interactions and magnetic field*, J. Stat.P. **40** (1985), 577–592. https://doi.org/10.1007/BF01017186
9. J. Vannimenus, *Modulated Phase of an Ising System with Competing Interactions on a Cayley Tree*, Z. Phys. B **43** (1981), 141–148. https://doi.org/10.1007/BF01293605

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