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ON UNIFORM STRONG LLN FOR WEIGHTED LÉVY-DRIVEN LINEAR PROCESS AND ITS APPLICATION TO STATISTICAL INFERENCE

In the article the averaged integral of the Lévy-driven linear process weighted by the complex exponential of a polynomial with real coefficients is considered. It is proved that uniformly over all real coefficients values of this polynomial such an averaged integral tends to zero a.s. It is also shown how the result obtained can be used to prove the LSE strong consistency of the chirp signal parameters.

1. Introduction

First of all, we will define a linear Lévy-driven stochastic process (see, for example, Ivanov, Leonenko, and Orlovskyi [8]). Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A Lévy process $L(t)$, $t \geq 0$, is a stochastic process on $(\Omega, \mathcal{F}, P)$ with independent and stationary increments, continuous in probability, with trajectories which are right-continuous with left limits (càdlàg) and $L(0) = 0$. For a general theory of Lévy processes we refer to Sato [12] and Applebaum [2].

Let (β, γ, Π) denote a characteristic triplet of the Lévy process $L(t)$, $t \in \mathbb{R}_+$, that is for all $t \in \mathbb{R}_+$

$$\ln \mathbb{E} \exp\{izL(t)\} = t\varkappa(z), \tag{1}$$

$$\varkappa(z) = i\beta z - \frac{1}{2}\gamma z^2 + \int\limits_{\mathbb{R}} \left(e^{izu} - 1 - iz\tau(u)\right) \Pi(du), z \in \mathbb{R}, \tag{2}$$

where $\beta \in \mathbb{R}$, $\gamma \geq 0$, and

$$\tau(u) = \begin{cases} u, |u| \leq 1; \\ \frac{u}{|u|}, |u| > 1. \end{cases}$$

The Lévy measure Π in (1) is a Radon measure on $\mathbb{R}\backslash\{0\}$ such that $\Pi(\{0\}) = 0$, and

$$\int\limits_{\mathbb{R}} \min(1, u^2)\Pi(du) < \infty.$$

The process $L(t)$ has finite qth moment for $q > 0$ ($\mathbb{E}|L(t)|^q < \infty$) if and only if

$$\int\limits_{|u|\geq 1} |u|^q \Pi(du) < \infty, \tag{3}$$

and $L(t)$ has finite pth exponential moment for $p > 0$ ($\mathbb{E}\exp\{pL(t)\} < \infty$) if and only if

$$\int\limits_{|u|\geq 1} e^{pu} \Pi(du) < \infty, \tag{4}$$

see, for example, Sato [12], Theorem 25.3.

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If $L(t)$, $t \in \mathbb{R}_+$, is a Lévy process with triplet (β, γ, Π), then the process $-L(t)$, $t \in \mathbb{R}_+$, is also a Lévy process with characteristics $(-\beta, \gamma, \widetilde{\Pi})$ with $\widetilde{\Pi} = \Pi(-A)$ for any Borel set A. Then we modify it to be càglàd (Anh, Heyde, and Leonenko [1]) and introduce a two-sided Lévy process $L(t)$, $t \in \mathbb{R}$, defined for $t < 0$ to be equal an independent copy of $-L(-t)$.

Let $\hat{a} : \mathbb{R} \to \mathbb{R}$ be a measurable function. Consider the linear stochastic process

$$\varepsilon(t) = \int_{\mathbb{R}} \hat{a}(t-s)dL(s), t \in \mathbb{R}, \tag{5}$$

assuming $\hat{a}$ to be a real valued function, and besides

$$\hat{a} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \text{ or } \hat{a} \in L_2(\mathbb{R}) \text{ with } \mathbb{E}L(1) = 0. \tag{6}$$

Under the conditions (6) and

$$\int_{\mathbb{R}} u^2 \Pi(du) < \infty,$$

the stochastic integral in (5) is well-defined in the sense of stochastic integration introduced by Rajput and Rosinski [10].

The popular choices for kernel $\hat{a}$ in (5) are Gamma type kernels:

(i) $\hat{a}(t) = t^{\alpha} e^{-\lambda t} \mathbb{1}_{[0,\infty)}(t)$, $\lambda > 0$, $\alpha > -\frac{1}{2}$;
(ii) $\hat{a}(t) = e^{-\lambda t} \mathbb{1}_{[0,\infty)}(t)$, $\lambda > 0$ (Ornstein-Uhlenbeck process);
(iii) $\hat{a}(t) = e^{-|\lambda| t}$, $\lambda > 0$ (well-balanced Ornstein-Uhlenbeck process).

In the text of the article it will be used the following condition.

A. $\mathbb{E}L(1) = 0$; $a = \sup_{t \in \mathbb{R}} |\hat{a}(t)| < \infty$; $||\hat{a}|| = \int_{\mathbb{R}} |\hat{a}(t)|dt < \infty$, and the measure Π satisfies (4) for some $p > 0$.

For arbitrary $q \in \mathbb{N}$ consider the family of polynomials

$$P_q(t) = b_1 t + b_2 t^2 + ... + b_q t^q, t \in \mathbb{R}_+, \tag{7}$$

with coefficients $b^{(q)} = (b_1, ..., b_q) \in \mathbb{R}^q$.

Theorem 1.1. *If the condition **A** is met, then for any* $q \in \mathbb{N}$

$$\xi_{qT} = \sup_{b^{(q)} \in \mathbb{R}^q} \left| T^{-1} \int_0^T \exp\left\{-iP_q(t)\right\} \varepsilon(t)dt \right| \to 0 \text{ a.s., as } T \to \infty. \tag{8}$$

The proof of such an uniform strong LLN for $q = 1$ and various stochastic processes ε has been obtained by many authors in connection with the problem of detecting hidden periodicities (see, for example, Ivanov et al. [9]). For $q = 2$ and Gaussian strongly or weakly dependent processes the uniform strong LLN is proved in Ivanov and Hladun [4] and this result is used to prove LSE strong consistency and asymptotic normality of the multiple chirp signal parameters (Ivanov and Hladun [4], [5]).

The proof of Theorem 1.1 is located in the Section 2. This theorem gives a positive answer to prof. A. Yu. Pilipenko question asked in the seminar "Malliavin Calculus and its Applications" during the authors report "Asymptotic properties of the LSE for chirp signal parameters" on March 12, 2024 (see on YouTube) on the proof of uniform strong LLN for polynomials $P_q(t)$ of orders $q > 2$.

In Section 3, two theorems are formulated on LSE strong consistency for multiple chirp signals parameters in the models with noise of the type (5), and their proofs are based significantly on Theorem 1.1 similarly to Ivanov and Hladun [4], [6].

2. Proof of Theorem 1.1

Introduce the notation

$$\varepsilon_q(t) = \prod_{B_r} \varepsilon\left(\sum_{p\in B_r} u_p + t\right), \tag{9}$$

where product occurs over all subsets B_r of the set $\{1, 2, ..., q\}$. Then

$$\mathbb{E}\varepsilon_q(t)\varepsilon_q(s) = \mathbb{E}\prod_{B_r} \varepsilon\left(\sum_{p\in B_r} u_p + t\right)\varepsilon\left(\sum_{p\in B_r} u_p + s\right), \tag{10}$$

and product in (10) contains 2^{q+1} factors.

Lemma 2.1. *Let the condition* ***A*** *be satisfied. Then for any* $q \in \mathbb{N}$

$$\mathbb{E}\xi^2_{qT} \le \prod_{i=0}^{q-1} 2^{2^{-i}}$$
$$\times\left(T^{-(q+2)}\underset{(u_1)}{\int_0^T}\underset{(u_2)}{\int_0^T}\dots\underset{(u_q)}{\int_0^T}\underset{(t)}{\int_0^T}\underset{(s)}{\int_0^T}|\mathbb{E}\varepsilon_q(t)\varepsilon_q(s)|dtdsdu_q...du_2du_1\right)^{2^{-q}}. \tag{11}$$

Proof. Let's carry out a detailed proof for $q = 3$. The proof for arbitrary $q \in \mathbb{N}$ is absolutely similar, but is much more cumbersome, and we will pay attention only to the key points of such a proof.

Consider the cubic polynomials $P_3(t) = b_1t+b_2t^2+b_3t^3$, $t \in \mathbb{R}_+$, $b^{(3)} = (b_1, b_2, b_3) \in \mathbb{R}^3$, and write down

$$\xi_{3T} = \sup_{b^{(3)}\in\mathbb{R}^3}\left|T^{-1}\int_0^T \exp\{-iP_3(t)\}\,\varepsilon(t)dt\right|. \tag{12}$$

Denote the expression under the supremum sign by η_{3T}. Then

$$\eta^2_{3T} = T^{-2}\int_0^T\int_0^T \exp\{-i(P_3(t) - P_3(s))\}\,\varepsilon(t)\varepsilon(s)dtds$$
$$= T^{-2}\iint_{t>s} + T^{-2}\iint_{t<s} = I_1 + I_2. \tag{13}$$

Making the change of variables $t - s = u_1$, $s = s'$, and then denoting s' again by the letter s, we get obviously

$$t^3 - s^3 = u_1^3 + 3u_1^2 s + 3u_1 s, \quad t^2 - s^2 = u_1^2 + 2u_1 s, \quad t - s = u_1.$$

Then

$$P_3(t) - P_3(s) = P_3(u_1) + 3b_3u_1s^2 + (3b_3u_1^2 + 2b_2u_1)s, \tag{14}$$

and

$$|I_1| = T^{-2}\left|\int_0^T \exp\{-iP_3(u_1)\}\left(\int_0^{T-u_1} \exp\left\{-i(3b_3u_1s^2+(3b_3u_1^2+2b_2u_1)s)\right\}\right.\right.$$
$$\left.\left.\times\,\varepsilon(s+u_1)\varepsilon(s)ds\right)du_1\right|$$

$$\leq T^{-2}\int_0^T\left|\int_0^{T-u_1}\exp\left\{-i(3b_3u_1s^2+(3b_3u_1^2+2b_2u_1)s)\right\}\varepsilon(s+u_1)\varepsilon(s)ds\right|du_1. \tag{15}$$

Renaming variables t to s, and s to t in the integral I_2 we obtain similarly

$$|I_2| = T^{-2}\left|\int_0^T \exp\{iP_3(u_1)\}\left(\int_0^{T-u_1} \exp\left\{i(3b_3u_1s^2+(3b_3u_1^2+2b_2u_1)s)\right\}\right.\right.$$
$$\left.\left.\times\,\varepsilon(s+u_1)\varepsilon(s)ds\right)du_1\right|$$

$$\leq T^{-2}\int_0^T\left|\int_0^{T-u_1}\exp\left\{i(3b_3u_1s^2+(3b_3u_1^2+2b_2u_1)s)\right\}\varepsilon(s+u_1)\varepsilon(s)ds\right|du_1. \tag{16}$$

Let's take $\varepsilon(s+u_1)\varepsilon(s)=\varepsilon_1(s)$. Then from (15) and (16) it follows

$$\mathbb{E}\xi_{3T}^2 \leq 2T^{-2}\int_0^T \mathbb{E}\sup_{(b_2,b_3)\in\mathbb{R}^2}\left|\int_0^{T-u_1}\exp\left\{-i(3b_3u_1s^2+(3b_3u_1^2+2b_2u_1)s)\right\}\varepsilon_1(s)ds\right|du_1$$
$$\leq 2T^{-2}\int_0^T\left(\mathbb{E}\sup_{(b_2,b_3)\in\mathbb{R}^2}\int_0^{T-u_1}\int_0^{T-u_1}\exp\left\{-i(3b_3u_1(t^2-s^2)+(3b_3u_1^2+2b_2u_1)(t-s))\right\}\right.$$
$$\left.\times\,\varepsilon_1(t)\varepsilon_1(s)dtds\right)^{\frac{1}{2}}du_1. \tag{17}$$

Rewrite double inner integral in (17) again as

$$\int_0^{T-u_1}\int_0^{T-u_1} = \iint_{t>s} + \iint_{t<s} = I_{21}+I_{22}. \tag{18}$$

Making the change of variables $t-s=u_2$, $s=s'\to s$ in the integral I_{21}, we obtain

$$3b_3u_1(t^2-s^2)+(3b_3u_1^2+2b_2u_1)(t-s) = 3b_3u_1(u_2^2+2u_2s)+(3b_3u_1^2+2b_2u_1)u_2,$$

that is

$$|I_{21}| = \left|\int_0^{T-u_1}\exp\left\{-i(3b_3u_1u_2^2+3b_3u_1^2u_2+2b_2u_1u_2)\right\}\right.$$
$$\left.\times\int_0^{T-u_1-u_2}\exp\{-i(6b_3u_1u_2s)\}\,\varepsilon_1(s+u_2)\varepsilon_1(s)dsdu_2\right|$$

$$\leq \int_0^{T-u_1} \left| \int_0^{T-u_1-u_2} \exp\left\{-i(6b_3u_1u_2s)\right\} \varepsilon_1(s+u_2)\varepsilon_1(s)ds \right| du_2. \tag{19}$$

Similarly,

$$|I_{22}| \leq \int_0^{T-u_1} \left| \int_0^{T-u_1-u_2} \exp\left\{i(6b_3u_1u_2s)\right\} \varepsilon_1(s+u_2)\varepsilon_1(s)ds \right| du_2. \tag{20}$$

Using notation $\varepsilon_2(s) = \varepsilon_1(s+u_2)\varepsilon_1(s)$, from (17)-(20) we arrive at the inequality

$$\begin{aligned} &\mathbb{E} \sup_{(b_2,b_3)\in\mathbb{R}^2} \int_0^{T-u_1}\int_0^{T-u_1} \exp\left\{-i(3b_3u_1(t^2-s^2) + (3b_3u_1^2+2b_2u_1)(t-s))\right\} \varepsilon_1(t)\varepsilon_1(s)dtds \\ &\leq 2\int_0^{T-u_1} \left(\mathbb{E}\sup_{b_3\in\mathbb{R}} \int_0^{T-u_1-u_2}\int_0^{T-u_1-u_2} \exp\left\{-i(6b_3u_1u_2(t-s))\right\}\varepsilon_2(t)\varepsilon_2(s)dtds \right)^{\frac{1}{2}} du_2. \end{aligned} \tag{21}$$

As before, we will make a change of variables $t-s=u_3$, $s=s' \to s$ in the double integral under the square root sign in the right hand side of (21):

$$\int_0^{T-u_1-u_2}\int_0^{T-u_1-u_2} \exp\left\{-i(6b_3u_1u_2(t-s))\right\}\varepsilon_2(t)\varepsilon_2(s)dtds = \iint_{t>s} + \iint_{t<s} = I_{31} + I_{32}. \tag{22}$$

We put $\varepsilon_3(s) = \varepsilon_2(s+u_3)\varepsilon_2(s)$ and get the following majorant:

$$\begin{aligned} \mathbb{E}\sup_{b_3\in\mathbb{R}} |I_{31}| &= \mathbb{E}\sup_{b_3\in\mathbb{R}} \left| \int_0^{T-u_1-u_2} \exp\left\{-i(6b_3u_1u_2u_3)\right\} \int_0^{T-u_1-u_2-u_3} \varepsilon_3(s)dsdu_3 \right| \\ &\leq \int_0^{T-u_1-u_2} \mathbb{E} \left| \int_0^{T-u_1-u_2-u_3} \varepsilon_3(s)ds \right| du_3 \\ &\leq \int_0^{T-u_1-u_2} \left(\int_0^{T-u_1-u_2-u_3}\int_0^{T-u_1-u_2-u_3} \mathbb{E}\varepsilon_3(t)\varepsilon_3(s)dtds \right)^{\frac{1}{2}} du_3. \end{aligned} \tag{23}$$

For $\mathbb{E}\sup_{b_3\in\mathbb{R}} |I_{32}|$ we get the same upper bound.

Collecting formulas (12)-(23) we derive the following inequality

$$\begin{aligned} \mathbb{E}\xi_{3T}^2 \leq 2T^{-2} \int_0^T \left(2\int_0^{T-u_1} \left(2\int_0^{T-u_1-u_2} \left(\int_0^{T-u_1-u_2-u_3}\int_0^{T-u_1-u_2-u_3} \mathbb{E}\varepsilon_3(t)\varepsilon_3(s)dtds \right)^{\frac{1}{2}} \right.\right. \\ \left.\left. \times\, du_3 \right)^{\frac{1}{2}} du_2 \right)^{\frac{1}{2}} du_1. \end{aligned} \tag{24}$$

Taking into account the notation $\varepsilon_3(t)$, $\varepsilon_2(t)$ and $\varepsilon_1(t)$ we get (see also (9))

$$\begin{aligned}\varepsilon_3(t) &= \varepsilon_2(t+u_3)\varepsilon_2(t) = \varepsilon_1(t+u_3+u_2)\varepsilon_1(t+u_3)\varepsilon_1(t+u_2)\varepsilon_1(t)\\ &= \varepsilon(u_1+u_2+u_3+t)\varepsilon(u_1+u_2+t)\varepsilon(u_1+u_3+t)\varepsilon(u_2+u_3+t)\\ &\times \varepsilon(u_1+t)\varepsilon(u_2+t)\varepsilon(u_3+t)\varepsilon(t).\end{aligned}$$

Thus in (24) the expectation

$$\begin{aligned}&\mathbb{E}\varepsilon_3(t)\varepsilon_3(s) = \mathbb{E}\varepsilon(u_1+u_2+u_3+t)\varepsilon(u_1+u_2+t)\varepsilon(u_1+u_3+t)\varepsilon(u_2+u_3+t)\\ &\times \varepsilon(u_1+t)\varepsilon(u_2+t)\varepsilon(u_3+t)\varepsilon(t)\varepsilon(u_1+u_2+u_3+s)\varepsilon(u_1+u_2+s)\varepsilon(u_1+u_3+s)\\ &\times \varepsilon(u_2+u_3+s)\varepsilon(u_1+s)\varepsilon(u_2+s)\varepsilon(u_3+s)\varepsilon(s)\end{aligned} \tag{25}$$

is the 16th mixed moment of the process ε values.

From (24) a rougher inequality follows:

$$\mathbb{E}\xi_{3T}^2 \le 2^{\frac{7}{4}} T^{-1}\int_0^T\left(T^{-1}\int_0^T\left(T^{-1}\int_0^T\left(T^{-2}\int_0^T\int_0^T|\mathbb{E}\varepsilon_3(t)\varepsilon_3(s)|dtds\right)^{\frac{1}{2}}du_3\right)^{\frac{1}{2}}du_2\right)^{\frac{1}{2}}du_1. \tag{26}$$

Making in (26) changes os variables $t\to Tt$, $s\to Ts$, $u_i\to Tu_i$, $i=1,2,3$, we obtain

$$\mathbb{E}\xi_{3T}^2 \le 2^{\frac{7}{4}}\int_0^1\left(\int_0^1\left(\int_0^1\left(\int_0^1\int_0^1|\mathbb{E}\varepsilon_3(Tt)\varepsilon_3(Ts)|dtds\right)^{\frac{1}{2}}du_3\right)^{\frac{1}{2}}du_2\right)^{\frac{1}{2}}du_1. \tag{27}$$

Let's apply to (27) Hölder's inequality

$$\int_0^1|f(x)|dx \le \left(\int_0^1|f(x)|^p dx\right)^{\frac{1}{p}}$$

3 times, first with respect to u_3 and $p=2$, next with respect to u_2 and $p=4$, and finally with respect to u_1 and $p=8$. In the end it turns out

$$\mathbb{E}\xi_{3T}^2 \le 2^{\frac{7}{4}}\left(\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1|\mathbb{E}\varepsilon_3(Tt)\varepsilon_3(Ts)|dtdsdu_3du_2du_1\right)^{\frac{1}{8}}. \tag{28}$$

Next we will sequentially make reverse changes of variables $Tt\to t$, $Ts\to s$, $Tu_i\to u_i$, $i=1,2,3$, and receive the inequality

$$\mathbb{E}\xi_{3T}^2 \le 2^{\frac{7}{4}}\left(T^{-5}\int_0^T\int_0^T\int_0^T\int_0^T\int_0^T|\mathbb{E}\varepsilon_3(t)\varepsilon_3(s)|dtdsdu_3du_2du_1\right)^{\frac{1}{8}}. \tag{29}$$

The question arises how to obtain an inequality (11) similar to (29) for an arbitrary $q\in\mathbb{N}$. The expression similar to the 1st row in (13) is

$$\eta_{qT}^2 = T^{-2}\int_0^T\int_0^T \exp\left\{-i(P_q(t)-P_q(s))\right\}\varepsilon(t)\varepsilon(s)dtds. \tag{30}$$

First of all we have to represent properly the differences t^q-s^q,...,t^4-s^4, to separate terms containing just powers of $t-s=u$ and terms containing both variables u and s. It will be a generalization of the relation (14). Note that $t^q-s^q = u\left((u+s)^{q-1}+(u+s)^{q-2}s+\ldots\right.$ $\left.+(u+s)s^{q-2}+s^{q-1}\right)$, and so on. This will lead to inequality similar to (17), where the coefficient b_1 is absent. Then we must take another $q-1$ steps to successively get rid of the coefficients b_2,...,b_q. In final analysis we will derive the inequalities similar to (26)-(29), and inequality (29) for general q to be of the form (11). □

The next step in the proof of Theorem 1.1 is to estimate the right hand side of the inequality (11).

From the condition **A** it follows (Anh, Heyde, and Leonenko [1]) for any $r \in \mathbb{N}$ (see (1), (2))

$$\ln \mathbb{E} \exp \left\{ i \sum_{j=1}^{r} z_j \varepsilon(t_j) \right\} = \int_{\mathbb{R}} \varkappa \left(\sum_{j=1}^{r} z_j \hat{a}(t_j - x) \right) dx. \tag{31}$$

In particular, from (31) it can be seen that ε is strictly stationary process. Denote by

$$m_r(t_1, ..., t_r) = \mathbb{E}\varepsilon(t_1)...\varepsilon(t_n),$$

$$c_r(t_1, ..., t_r) = i^{-r} \frac{\partial^r}{\partial z_1 ... \partial z_r} \ln \mathbb{E} \exp \left\{ i \sum_{j=1}^{r} z_j \varepsilon(t_j) \right\} \Bigg|_{z_1 = ... = z_r = 0} \tag{32}$$

the moment and cumulant functions correspondingly of order r of the process ε.

The explicit expression for cumulants of the stochastic process ε can be obtained from (31) by direct calculations:

$$c_r(t_1, ..., t_r) = d_r \int_{\mathbb{R}} \prod_{j=1}^{r} \hat{a}(t_j - x) dx, \tag{33}$$

where d_r is the rth cumulant of the random variable $L(1)$. From equation (1), taking $t = 1$, one can find under condition **A**: $d_1 = 0$, $d_2 = \mathbb{E}L^2(1) = -\varkappa''(0)$, $d_3 = \mathbb{E}L^3(1)$, $d_4 = \mathbb{E}L^4(1) - 3(\mathbb{E}L^2(1))^2$, and so on. Note also that $m_2(t_1, t_2) = c_2(t_1, t_2) = B(t_1 - t_2)$, where

$$B(t) = d_2 \int_{\mathbb{R}} \hat{a}(t + x)\hat{a}(x) dx, t \in \mathbb{R}. \tag{34}$$

Let $I = \{1, 2, ..., Q\}$, $I_p = \{i_1, ..., i_{l_p}\} \subset I$, $c(I_p) = c_{l_p}\left(t_{i_1}, ..., t_{i_{l_p}}\right)$, $m(I) = m_Q(t_1, ..., t_Q)$. Then the following Leonov-Shiryaev formula is valid (see, for example, Ivanov and Leonenko [7])

$$m(I) = \sum_{A_r} \prod_{p=1}^{r} c(I_p), \tag{35}$$

where $\sum\limits_{A_r}$ denotes summation over all unordered partitions $A_r = \left\{ \bigcup\limits_{p=1}^{r} I_p \right\}$ of the set I into the sets I_1,...,I_r such that $I = \bigcup\limits_{p=1}^{r} I_p$, $I_i \cap I_j = \emptyset$, $i \neq j$.

Lemma 2.2. *The fulfillment of condition **A** entails for any $q \in \mathbb{N}$ the relation*

$$\mathbb{E}\xi_{qT}^2 = O(T^{-2^{-q}}), \text{ as } T \to \infty. \tag{36}$$

Proof. As before, we will consider in detail the proof for $q = 3$ and apply the formula (35) for $Q = 2^{q+1} = 16$ to integrand expression (25) in (29), namely: the 16th mixed moment of the process ε values can be represented as a sum of products of the process ε cumulants, which, in turn, can be divided into 55 sets in accordance with the orders of cumulants in these products (see Appendix 1). Each product of cumulants in each of these 55 sets, using condition **A**, can be estimated in a special way. Bellow we give an example how this should be done.

Example 2.1. Consider the product $c_6()c_4()c_3()c_3()$ and use the formula (33). Then we can estimate the 6th cumulant as $|c_6()| \le |d_6|a^5||\hat{a}||$. Similarly $|c_4()| \le |d_4|a^3||\hat{a}||$, for the 1st cumulant of the 3rd order $|c_3()| \le |d_3|a^2||\hat{a}||$. The second cumulant of the 3rd order (the last cumulant of the given product) can be bounded as

$$|c_3()| \le |d_3| \int\limits_{\mathbb{R}} |\hat{a}(... - x)\hat{a}(... - x)\hat{a}(... - x)|dx \le |d_3|a \int\limits_{\mathbb{R}} |\hat{a}(... - x)||\hat{a}(... - x)|dx, \tag{37}$$

where 3 dots mean some sums of variables from formula (25). In the 1st and 2nd factors under the integral sign in the right hand side of (37) the variable sets V_1 and V_2 from the set $V = \{t, s, u_1, u_2, u_3\}$ do not coincide. If $V_1 \subset V_2$, then we just rearrange the factors. In any case in the old or new the 1st factors we can specify a variable that is missing in the 2nd factor. Suppose, for example, the 1st and the 2nd factors are of the form $\hat{a}(... + t - x)$ and $\hat{a}(... + s - x)$ correspondently. Then by Fubini theorem

$$\int\limits_0^T \left(\int\limits_{\mathbb{R}} |\hat{a}(... + t - x)||\hat{a}(... + s - x)|dx \right) dt \le ||\hat{a}|| \int\limits_{\mathbb{R}} |\hat{a}(... + s - x)|dx \le ||\hat{a}||^2, \tag{38}$$

and therefore the fivefold integral in (29) corresponding to the given product of cumulants can be bounded by the value $|d_6 d_4|d_3^2 a^{11}||\hat{a}||^5 T^4$ (see the 35th row in Appendix 1).

Let $d = \max\limits_i |d_i|$, $i = \{2, 3, ..., 2^{q+1} - 2, 2^{q+1}\}$, for arbitrary $q \in \mathbb{N}$. If $q = 3$, then $d = \max\limits_i |d_i|$, $i = \{2, 3, ..., 14, 16\}$. Let's majorize in each row of the Appendix 1 table each value $|d_i|$ by d. After this one can notice that all the estimates on the right side of the Appendix 1 table split into $2^3 = 8$ types:

$$d^{j-1}a^{16-j}||\hat{a}||^j T^4, \quad j = \overline{2, 9}. \tag{39}$$

The terms (39) are present in the estimate of integral in formula (29) with some integer coefficients, and their calculation for $q = 3$ is not included in our plans. Instead, we are going to write a rougher but manageable bound by counting the total number of terms in formula (35) provided $d_1 = 0$. Generally speaking, the sum (35) contains B_Q terms, where B_Q is the Bell number, i.e. the number of all possible unordered partitions of an Q-element set (see, for example, Rota [11]).

Let B_m, $m \ge 1$, be the Bell numbers, S_m be the numbers of unordered partitions of the set $\{1, 2, ..., m\}$ into subsets that do not contain singletons. Put by definition $S_0 = 1$. Then

$$B_m = \sum_{j=0}^{m} C_m^j S_{m-j}, \quad S_m = B_m - \sum_{j=1}^{m} C_m^j S_{m-j}. \tag{40}$$

Using recurrence relation (40) we write the first $Q = 16$ numbers S_m in Appendix 2 to serve the case $q = 3$.

Thus, taking into account the above reasoning, we can write the following bound:

$$\begin{gathered} \mathbb{E}\xi_{3T}^2 \le 2^{\frac{7}{4}} \left(S_{16} \sum_{j=2}^{9} \left(d^{j-1}a^{16-j}||\hat{a}||^j \right) \right)^{\frac{1}{8}} T^{-\frac{1}{8}}, \\ S_{16} = 1\ 216\ 070\ 380, \quad \sqrt[8]{S_{16}} \approx 13.665 \text{ (see Appendix 2).} \end{gathered} \tag{41}$$

Passing to an arbitrary $q \in \mathbb{N}$ and subjecting formulas (10), (11) to similar processing, we obtain the general rough but observable inequality proving Lemma 2.2:

$$\mathbb{E}\xi_{qT}^2 \le \prod_{i=0}^{q-1} 2^{2^{-i}} \left(S_{2^{q+1}} \sum_{j=2}^{2^q+1} \left(d^{j-1} a^{2^{q+1}-j} ||\hat{a}||^j \right) \right)^{2^{-q}} T^{-2^{-q}}. \tag{42}$$

□

In particular, the last inequality is correct for $q = 2$ as well, namely:

$$\mathbb{E}\xi_{2T}^2 \le 2^{\frac{3}{2}} \left(S_8 \sum_{j=2}^{5} \left(d^{j-1} a^{8-j} ||\hat{a}||^j \right) \right)^{\frac{1}{4}} T^{-\frac{1}{4}}, \quad S_8 = 715. \tag{43}$$

However in the next section of the paper we will offer a more accurate estimate for $q = 2$.

The last part of Theorem 1.1 proof is standard (see, for example, Ivanov and Hladun [4]). Returning to inequality (42) we take $T_n = n^\alpha$ with number $\alpha > 2^q$. Then $\sum_{n=1}^{\infty} \mathbb{E}\xi_{qT_n}^2 < \infty$, and $\xi_{qT_n} \to 0$ a.s., as $n \to \infty$. Consider the sequence of random variables

$$\begin{aligned}
\zeta_n &= \sup_{T_n \le T < T_{n+1}} |\xi_{qT} - \xi_{qT_n}| \\
&= \sup_{T_n \le T < T_{n+1}} \left| \sup_{b^{(q)} \in \mathbb{R}^q} \left| T^{-1} \int_0^T \exp\{-iP_q(t)\}\varepsilon(t)dt \right| - \sup_{b^{(q)} \in \mathbb{R}^q} \left| T_n^{-1} \int_0^{T_n} \exp\{-iP_q(t)\}\varepsilon(t)dt \right| \right| \\
&\le \sup_{T_n \le T < T_{n+1}} \sup_{b^{(q)} \in \mathbb{R}^q} \left| T^{-1} \int_0^T \exp\{-iP_q(t)\}\varepsilon(t)dt - T_n^{-1} \int_0^{T_n} \exp\{-iP_q(t)\}\varepsilon(t)dt \right| \\
&\le \sup_{T_n \le T < T_{n+1}} \left[\sup_{b^{(q)} \in \mathbb{R}^q} \left| (T^{-1} - T_n^{-1}) \int_0^{T_n} \exp\{-iP_q(t)\}\varepsilon(t)dt \right| \right. \\
&\qquad \left. + \sup_{b^{(q)} \in \mathbb{R}^q} \left| T^{-1} \int_{T_n}^{T} \exp\{-iP_q(t)\}\varepsilon(t)dt \right| \right] \\
&\le \frac{T_{n+1} - T_n}{T_n} \xi_{qT_n} + T_n^{-1} \int_{T_n}^{T_{n+1}} |\varepsilon(t)|dt = \zeta_{n1} + \zeta_{n2}.
\end{aligned}$$

Obviously, $\zeta_{n1} \to 0$ a.s., as $n \to \infty$. On the other hand,

$$\mathbb{E}\zeta_{n2}^2 = T_n^{-2} \int_{T_n}^{T_{n+1}} \int_{T_n}^{T_{n+1}} \mathbb{E}|\varepsilon(t)\varepsilon(s)|dtds \le B(0) \left(\frac{T_{n+1} - T_n}{T_n} \right)^2 = O(n^{-2}),$$

since according to condition **A** and formula (34) $B(0) \le d_2 a ||\hat{a}||$. Thus, $\sum_{n=1}^{\infty} \mathbb{E}\zeta_{n2}^2 \le \infty$, and $\zeta_{n2} \to 0$ a.s., as $n \to \infty$.

3. Some statistical applications for $q = 2$

First of all, we will clarify inequality (43). Application of formula (35) to (10) and (11) for $q = 2$ gives us the following sum of cumulants:

1) $c_8()$ – 1 term; 2) $c_6()c_2()$ – 28 terms; 3) $c_5()c_3()$ – 56 terms;

4) $c_4()c_4()$ – 35 terms; 5) $c_4()c_2()c_2()$ – 210 terms; 6) $c_3()c_3()c_2()$ – 280 terms; 7) $c_2()c_2()c_2()c_2()$ – 105 terms.

Using the notation introduced above, we get

$$\mathbb{E}\xi_{2T}^2 \le H_1 T^{-\frac{1}{4}}, \tag{44}$$

$$H_1 = 2^{\frac{3}{2}}\Big(|d_8|a^6||\hat{a}||^2 + 28|d_6|d_2 a^5||\hat{a}||^3 + 56|d_5 d_3|a^5||\hat{a}||^3 + 35 d_4^2 a^5||\hat{a}||^3$$
$$+ 210|d_4|d_2^2 a^4||\hat{a}||^4 + 280 d_3^2 d_2 a^4||\hat{a}||^4 + 105 d_2^4 a^3||\hat{a}||^5\Big)^{\frac{1}{4}}$$
$$\le 2^{\frac{3}{2}}\left(da^6||\hat{a}||^2 + 119 d^2 a^5||\hat{a}||^3 + 490 d^3 a^4||\hat{a}||^4 + 105 d^4 a^3||\hat{a}||^5\right)^{\frac{1}{4}} = H_2.$$

So,

$$\mathbb{E}\xi_{2T}^2 \le H_2 T^{-\frac{1}{4}}, \tag{45}$$

and moreover $1 + 119 + 490 + 105 = 715 = S_8$, that is H_2 is less than correspondent constant in (43). Unfortunately for $q \ge 3$ such an improvement of the constant in inequality (42) is too complicated if we use this method of Theorem 1.1 proof.

Next we are going to show how Theorem 1.1 can be used to prove the strong consistency of LSE of multiple chirp signal parameters (see Ivanov and Hladun [4], [5], [6]).

Suppose we observe a stochastic process

$$X(t) = g(t, \theta^0) + \varepsilon(t), \quad t \in \mathbb{R}_+, \tag{46}$$

where

$$g(t, \theta^0) = \sum_{j=1}^{N}\left(A_j^0 \cos\left(\phi_j^0 t + \psi_j^0 t^2\right) + B_j^0 \sin\left(\phi_j^0 t + \psi_j^0 t^2\right)\right), \tag{47}$$

$$\theta^0 = \left(A_1^0, B_1^0, \phi_1^0, \psi_1^0, ..., A_N^0, B_N^0, \phi_N^0, \psi_N^0\right), \tag{48}$$

$\left(A_j^0\right)^2 + \left(B_j^0\right)^2 > 0$, $j = \overline{1,N}$; $\varepsilon = \{\varepsilon(t), t \in \mathbb{R}\}$ is a Lévy-driven linear process described in the Section 1 of the paper. Assuming that the true values of amplitudes A_j^0, B_j^0, $j = \overline{1,N}$, are different numbers and the true values of frequencies ϕ_j^0, $j = \overline{1,N}$, and chirp rates ψ_j^0, $j = \overline{1,N}$, are different positive numbers, we arrange the chirp rates $\psi^0 = \left(\psi_1^0, ..., \psi_N^0\right)$ in increasing order and suppose

$$\psi^0 \in \Psi(\underline{\psi}, \overline{\psi}) = \left\{\psi = (\psi_1, ..., \psi_N) \in \mathbb{R}^N : 0 \le \underline{\psi} < \psi_1 < ... < \psi_N < \overline{\psi} < +\infty\right\}.$$

In turn, we introduce also the parametric set

$$\Phi(\underline{\phi}, \overline{\phi}) = \left\{\phi = (\phi_1, ..., \phi_N) : 0 \le \underline{\phi} < \phi_j < \overline{\phi} < +\infty, j = \overline{1,N}\right\},$$

and $\phi^0 = \left(\phi_1^0, ..., \phi_N^0\right) \in \Phi(\underline{\phi}, \overline{\phi})$.

Consider monotonically non-decreasing family of open sets $\Psi_T \subset \Psi\left(\underline{\psi}, \overline{\psi}\right), T > T_0 > 0$, containing vector ψ^0, such that $\bigcup_{T > T_0} \Psi_T = \widetilde{\Psi}, \widetilde{\Psi}^c = \Psi^c\left(\underline{\psi}, \overline{\psi}\right)$, with the following properties

B. 1) $\inf_{\substack{1 \le j \le N-1 \\ \psi \in \Psi_T}} T^2\left(\psi_{j+1} - \psi_j\right) \to +\infty$, as $T \to \infty$;

2) $\inf_{\psi \in \Psi_T} T^2 \psi_1 \to +\infty$, as $T \to \infty$.

Definition 3.1. Any random vector

$$\theta_T = \left(A_{1T}, B_{1T}, \phi_{1T}, \psi_{1T}, ..., A_{NT}, B_{NT}, \phi_{NT}, \psi_{NT}\right) \tag{49}$$

such that it is a point of the functional

$$Q_T(\theta) = T^{-1}\int_0^T [X(t) - g(t,\theta)]^2\, dt \tag{50}$$

absolute minimum on the parametric set $\Theta_T^c \subset \mathbb{R}^{4N}$, where amplitudes A_j,B_j,$j=\overline{1,N}$, can take any values and parameters (ϕ,ψ) take values in the set $\Phi^c(\underline{\phi},\overline{\phi})\times\Psi_T^c, T>T_0>0$, is called LSE of the parameter θ^0.

Theorem 3.1. *Let the conditions* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be satisfied. Then LSE* θ_T *is a strongly consistent estimate of vector parameter* θ^0*, namely:* $A_{jT}\to A_j^0$, $B_{jT}\to B_j^0$, $T\left(\phi_{jT}-\phi_j^0\right)\to 0$, $T^2\left(\psi_{jT}-\psi_j^0\right)\to 0$ *a.s., as* $T\to\infty$, $j=\overline{1,N}$.

Proof. The use of Theorem 1.1 for $q=2$ and $(b_1,b_2)=(\phi,\psi)$ leads to the fact that the proof of Theorem 3.1 does not differ from the proof of Theorem 1 in Ivanov and Hladun [4]. □

Consider again the observation model of the type (46) with the random noise ε from the Section 1 but with another regression function

$$g(t,\theta^0) = \sum_{j=1}^{N}\left(A_j^0\cos(\phi_j^0 t) + B_j^0\sin(\phi_j^0 t) + C_j^0\cos(\psi_j^0 t^2) + D_j^0\sin(\psi_j^0 t^2)\right). \tag{51}$$

Statistical model of the type (46) with function (51) is said to be a multiple chirp-like signal (see Grover, Kundu, and Mitra [3]) with unknown vector parameter

$$\theta^0 = \left(A_1^0, B_1^0, \phi_1^0, C_1^0, D_1^0, \psi_1^0, ..., A_N^0, B_N^0, \phi_N^0, C_N^0, D_N^0, \psi_N^0\right), \tag{52}$$

$\left(A_j^0\right)^2+\left(B_j^0\right)^2>0$, $\left(C_j^0\right)^2+\left(D_j^0\right)^2>0$, $j=\overline{1,N}$. Let us assume that the amplitudes A_j^0, B_j^0, C_j^0, D_j^0, $j=\overline{1,N}$, are different numbers and the frequencies ϕ_j^0, $j=\overline{1,N}$, and chirp rates ψ_j^0, $j=\overline{1,N}$, are different positive numbers that forms monotonically increasing sequences. For some fixed numbers $0\le\underline{\phi}<\overline{\phi}<+\infty$, $0\le\underline{\psi}<\overline{\psi}<+\infty$ consider the sets

$$\Psi(\underline{\psi},\overline{\psi}) = \left\{\psi=(\psi_1,...,\psi_N)\in\mathbb{R}^N : \underline{\psi}<\psi_1<...<\psi_N<\overline{\psi}\right\},$$

$$\Phi(\underline{\phi},\overline{\phi}) = \left\{\phi=(\phi_1,...,\phi_N)\in\mathbb{R}^N : \underline{\phi}<\phi_1<...<\phi_N<\overline{\phi}\right\}, \tag{53}$$

such that $\phi^0=(\phi_1,...,\phi_N)\in\Phi(\underline{\phi},\overline{\phi})$, $\psi^0=(\psi_1,...,\psi_N)\in\Psi(\underline{\psi},\overline{\psi})$.

Consider monotonically non-decreasing families of open sets $\Phi_T\subset\Phi\left(\underline{\phi},\overline{\phi}\right)$, $\Psi_T\subset\Psi\left(\underline{\psi},\overline{\psi}\right)$, $T>T_0>0$, containing vectors ϕ^0,ψ^0, such that $\left(\bigcup\limits_{T>T_0}\Phi_T\right)^c=\Phi^c\left(\underline{\phi},\overline{\phi}\right)$, $\left(\bigcup\limits_{T>T_0}\Psi_T\right)^c=\Psi^c\left(\underline{\psi},\overline{\psi}\right)$, with the following properties

C. 1) $\inf\limits_{\substack{1\le j\le N-1\\ \phi\in\Phi_T^c}} T\left(\phi_{j+1}-\phi_j\right),\ \inf\limits_{\phi\in\Phi_T^c} T\phi_1\to+\infty$, as $T\to\infty$;

2) $\inf\limits_{\substack{1\le j\le N-1\\ \psi\in\Psi_T^c}} T^2\left(\psi_{j+1}-\psi_j\right),\ \inf\limits_{\psi\in\Psi_T^c} T^2\psi_1\to+\infty$, as $T\to\infty$.

Definition 3.2. Any random vector

$$\theta_T = (A_{1T}, B_{1T}, \phi_{1T}, C_{1T}, D_{1T}, \psi_{1T}, ..., A_{NT}, B_{NT}, \phi_{NT}, C_{NT}, D_{NT}, \psi_{NT}), \tag{54}$$

that minimizes the functional (50) with function $g(t,\theta)$ given by (51), on the parametric set $\Theta_T^c\subset\mathbb{R}^{6N}$, where amplitudes A_j, B_j, C_j, D_j, $j=\overline{1,N}$, can take any values and parameters $\phi_j, \psi_j, j = \overline{1, N}$, take values in the set $\Phi_T^c \times \Psi_T^c, T > T_0 > 0$, is called LSE of the parameter θ^0 given by (52).

Theorem 3.2. *Let the conditions **A** and **C** be satisfied. Then LSE θ_T is a strongly consistent estimate of parameter (52) in the sense that $A_{jT} \to A_j^0$, $B_{jT} \to B_j^0$, $T\left(\phi_{jT} - \phi_j^0\right) \to 0$, $C_{jT} \to C_j^0$, $D_{jT} \to D_j^0$, $T^2\left(\psi_{jT} - \psi_j^0\right) \to 0$ a.s., as $T \to \infty, j = \overline{1, N}$.*

Proof. We again use Theorem 1.1 for $q = 2$ and $(b_1, b_2) = (\phi, \psi)$. Then the proof of Theorem 3.2 is the same as proof of Theorem 1 of the paper Ivanov and Hladun [6]. □

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Appendix 1. Majorants for integrals of cumulants products in the case of the polynomials $P_3(t)$

№	Product of cumulants	Majorants
1.	$c_{16}()$	$\lvert d_{16}\rvert a^{14}\lvert\lvert\hat{a}\rvert\rvert^2T^4$
2.	$c_{14}()c_2()$	$\lvert d_{14}\rvert d_2a^{13}\lvert\lvert\hat{a}\rvert\rvert^3T^4$
3.	$c_{13}()c_3()$	$\lvert d_{13}d_3\rvert a^{13}\lvert\lvert\hat{a}\rvert\rvert^3T^4$
4.	$c_{12}()c_4()$	$\lvert d_{12}d_4\rvert a^{13}\lvert\lvert\hat{a}\rvert\rvert^3T^4$
5.	$c_{12}()c_2()c_2()$	$\lvert d_{12}\rvert d_2^2a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
6.	$c_{11}()c_5()$	$\lvert d_{11}d_5\rvert a^{13}\lvert\lvert\hat{a}\rvert\rvert^3T^4$
7.	$c_{11}()c_3()c_2()$	$\lvert d_{11}d_3\rvert d_2a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
8.	$c_{10}()c_6()$	$\lvert d_{10}d_6\rvert a^{13}\lvert\lvert\hat{a}\rvert\rvert^3T^4$
9.	$c_{10}()c_4()c_2()$	$\lvert d_{10}d_4\rvert d_2a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
10.	$c_{10}()c_3()c_3()$	$\lvert d_{10}\rvert d_3^2a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
11.	$c_{10}()c_2()c_2()c_2()$	$\lvert d_{10}\rvert d_2^3a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
12.	$c_9()c_7()$	$\lvert d_9d_7\rvert a^{13}\lvert\lvert\hat{a}\rvert\rvert^3T^4$
13.	$c_9()c_5()c_2()$	$\lvert d_9d_5\rvert d_2a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
14.	$c_9()c_4()c_3()$	$\lvert d_9d_4d_3\rvert a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
15.	$c_9()c_3()c_2()c_2()$	$\lvert d_9d_3\rvert d_2^2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
16.	$c_8()c_8()$	$d_8^2a^{13}\lvert\lvert\hat{a}\rvert\rvert^3T^4$
17.	$c_8()c_6()c_2()$	$\lvert d_8d_6\rvert d_2a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
18.	$c_8()c_5()c_3()$	$\lvert d_8d_5d_3\rvert a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
19.	$c_8()c_4()c_4()$	$\lvert d_8\rvert d_4^2a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
20.	$c_8()c_4()c_2()c_2()$	$\lvert d_8d_4\rvert d_2^2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
21.	$c_8()c_3()c_3()c_2()$	$\lvert d_8\rvert d_3^2d_2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
22.	$c_8()c_2()c_2()c_2()c_2()$	$\lvert d_8\rvert d_2^4a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
23.	$c_7()c_7()c_2()$	$d_7^2d_2a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
24.	$c_7()c_6()c_3()$	$\lvert d_7d_6d_3\rvert a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
25.	$c_7()c_5()c_4()$	$\lvert d_7d_5d_4\rvert a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
26.	$c_7()c_5()c_2()c_2()$	$\lvert d_7d_5\rvert d_2^2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
27.	$c_7()c_4()c_3()c_2()$	$\lvert d_7d_4d_3\rvert d_2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
28.	$c_7()c_3()c_3()c_3()$	$\lvert d_7\rvert\lvert d_3\rvert^3a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
29.	$c_7()c_3()c_2()c_2()c_2()$	$\lvert d_7d_3\rvert d_2^3a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
30.	$c_6()c_6()c_4()$	$d_6^2\lvert d_4\rvert a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
31.	$c_6()c_6()c_2()c_2()$	$d_6^2d_2^2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
32.	$c_6()c_5()c_5()$	$\lvert d_6\rvert d_5^2a^{12}\lvert\lvert\hat{a}\rvert\rvert^4T^4$
33.	$c_6()c_5()c_3()c_2()$	$\lvert d_6d_5d_3\rvert d_2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
34.	$c_6()c_4()c_4()c_2()$	$\lvert d_6\rvert d_4^2d_2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
35.	$c_6()c_4()c_3()c_3()$	$\lvert d_6d_4\rvert d_3^2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
36.	$c_6()c_4()c_2()c_2()c_2()$	$\lvert d_6d_4\rvert d_2^3a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
37.	$c_6()c_3()c_3()c_2()c_2()$	$\lvert d_6\rvert d_3^2d_2^2a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
38.	$c_6()c_2()c_2()c_2()c_2()c_2()$	$\lvert d_6\rvert d_2^5a^9\lvert\lvert\hat{a}\rvert\rvert^7T^4$
39.	$c_5()c_5()c_4()c_2()$	$d_5^2\lvert d_4\rvert d_2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
40.	$c_5()c_5()c_3()c_3()$	$d_5^2d_3^2a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
41.	$c_5()c_5()c_2()c_2()c_2()$	$d_5^2d_2^3a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
42.	$c_5()c_4()c_4()c_3()$	$\lvert d_5\rvert d_4^2\lvert d_3\rvert a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
43.	$c_5()c_4()c_3()c_2()c_2()$	$\lvert d_5d_4d_3\rvert d_2^2a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
44.	$c_5()c_3()c_3()c_3()c_2()$	$\lvert d_5d_3^3\rvert d_2a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
45.	$c_5()c_3()c_2()c_2()c_2()c_2()$	$\lvert d_5d_3\rvert d_2^4a^9\lvert\lvert\hat{a}\rvert\rvert^7T^4$
46.	$c_4()c_4()c_4()c_4()$	$d_4^4a^{11}\lvert\lvert\hat{a}\rvert\rvert^5T^4$
47.	$c_4()c_4()c_4()c_2()c_2()$	$\lvert d_4\rvert^3d_2^2a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
48.	$c_4()c_4()c_3()c_3()c_2()$	$d_4^2d_3^2d_2a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
49.	$c_4()c_4()c_2()c_2()c_2()c_2()$	$d_4^2d_2^4a^9\lvert\lvert\hat{a}\rvert\rvert^7T^4$
50.	$c_4()c_3()c_3()c_3()c_3()$	$\lvert d_4\rvert d_3^4a^{10}\lvert\lvert\hat{a}\rvert\rvert^6T^4$
51.	$c_4()c_3()c_3()c_2()c_2()c_2()$	$\lvert d_4\rvert d_3^2d_2^3a^9\lvert\lvert\hat{a}\rvert\rvert^7T^4$
52.	$c_4()c_2()c_2()c_2()c_2()c_2()c_2()$	$\lvert d_4\rvert d_2^6a^8\lvert\lvert\hat{a}\rvert\rvert^8T^4$
53.	$c_3()c_3()c_3()c_3()c_2()c_2()$	$d_3^4d_2^2a^9\lvert\lvert\hat{a}\rvert\rvert^7T^4$
54.	$c_3()c_3()c_2()c_2()c_2()c_2()c_2()$	$d_3^2d_2^5a^8\lvert\lvert\hat{a}\rvert\rvert^8T^4$
55.	$c_2()c_2()c_2()c_2()c_2()c_2()c_2()c_2()$	$d_2^8a^7\lvert\lvert\hat{a}\rvert\rvert^9T^4$

Appendix 2. The first 16 numbers S_m

m	The Bell numbers B_m	The numbers S_m
1.	1	0
2.	2	1
3.	5	1
4.	15	4
5.	52	11
6.	203	41
7.	877	162
8.	4 140	715
9.	21 147	3 425
10.	115 975	17 722
11.	678 570	98 263
12.	4 213 597	580 317
13.	27 644 437	3 633 280
14.	190 899 322	24 011 157
15.	1 382 958 545	166 888 165
16.	10 480 142 147	1 216 070 380

References

1. V. V. Anh, C. C. Heyde, and N. N. Leonenko, *Dynamic models of long-memory processes driven by Lévy noise*, Journal of Applied Probability **39** (2002), no. 4, 730–747. https://doi.org/10.1239/jap/1037816015
2. D. Applebaum, *Lévy Processes and stochastic calculus*, 2nd ed., Cambridge studies in advanced mathematics, Cambridge University Press, 2009. https://doi.org/10.1017/CBO9780511809781
3. R. Grover, D. Kundu, and A. Mitra, *Asymptotic properties of least squares estimators and sequential least squares estimators of a chirp-like signal model parameters*, Circuits Syst Signal Process **40** (2021), 5421–5465. https://doi.org/10.1007/s00034-021-01724-7
4. A. V. Ivanov and V. V. Hladun, *Consistency of the LSE for chirp signal parameters in the models with strongly and weakly dependent noise*, Austrian Journal of Statistics **52 (SI)** (2023), 107–126. https://doi.org/10.17713/ajs.v52iSI.1762
5. A. V. Ivanov and V. V. Hladun, *Asymptotic normality of the LSE for chirp signal parameters*, Modern Stochastics: Theory and Applications **11** (2024), no. 2, 195–216. https://doi.org/10.15559/24-VMSTA247
6. A. V. Ivanov and V. V. Hladun, *Asymptotic properties of the LSE for chirp-like signal parameters*, Austrian Journal of Statistics **54** (2025), no. 1, 33–60. (To appear)
7. A. V. Ivanov and N. N. Leonenko, *Statistical analysis of random fields*, Springer Dordrecht, 1989. https://doi.org/10.1007/978-94-009-1183-3
8. A. V. Ivanov, N. N. Leonenko, and I. V. Orlovskyi, *On the Whittle estimator for linear random noise spectral density parameter in continuous-time nonlinear regression models*, Stat Inference Stoch Process **23** (2020), 129–169. https://doi.org/10.1007/s11203-019-09206-z
9. A. V. Ivanov, N. N. Leonenko, M. D. Ruiz-Medina, and B. M. Zhurakovsky, *Estimation of harmonic component in regression with cyclically dependent errors*, Statistics **49** (2014), no. 1, 156–186. https://doi.org/10.1080/02331888.2013.864656
10. B. S. Rajput and J. Rosinski, *Spectral representations of infinitely divisible processes*, Probab. Th. Rel. Fields **82** (1989), 451–487. https://doi.org/10.1007/BF00339998
11. Gian-Carlo Rota, *The number of partitions of a set*, The American Mathematical Monthly **71** (1964), no. 5, 498–504. https://doi.org/10.1080/00029890.1964.11992270
12. K. Sato, *Lévy Processes and infinitely divisible distributions*, Cambridge studies in advanced mathematics, Cambridge University Press, 1999. https://books.google.com.ua/books?id=tbZPLquJjSoC

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