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PROPERTIES OF THE VERTEX OF A CONVEX HULL GENERATED BY A POISSON POINT PROCESS INSIDE A PARABOLA

A convex hull generated by the implementation of a Poisson point process inside a parabola is considered in the article. At that, the measure of intensity of the Poisson law is related to regularly varying functions near the boundary of the support. It is proven that the domain bounded by the perimeters of the convex hull and the boundary of the support - a parabola, can be represented as a sum of independent identically distributed random variables. Moreover, this value does not depend on the vertices of the convex hull itself. It is worth noting that having approximated the binomial point process by a Poisson one, P.Groeneboom [6], A.J.Cabo and P.Groeneboom [3], I.Hueter [9], T.Hsing [8] and others, using the martingale properties of stationary vertex processes, proved various options of the central limit theorem for functionals of a random convex hull in the case when the original distribution is uniformly concentrated in a convex polygon or ellipse. In this paper, the exact distribution and conditional distribution of vertex processes are found when the convex hull is generated by a inhomogeneous Poisson point process inside a parabola. In some special cases, it is shown that the area between the perimeter of the convex hull and the support of the distribution is expressed by the sum of independent random variables.

1. Introduction

Asymptotic analysis of order statistics is important in estimating unknown parameters of distribution and in determining the critical domain in testing statistical hypotheses. In particular, if the boundaries of the domain depend on the unknown estimated parameters, then the estimates are constructed using the extreme terms of the variation series, and they are consistent, asymptotically unbiased estimates and sufficient statistics. The convex hull is the most complete multivariate analog of extreme observations of a sample, in particular, if the support of a uniform sample is a convex domain, then the convex hull for estimating the support of the distribution is a consistent, asymptotically unbiased estimate and sufficient statistics, as in the one-dimensional case.

This research is devoted to the study of the properties of convex hulls generated by the implementation of a homogeneous Poisson process on a plane inside the parabola. It is worth noting that the field of study of the convex hull relates to stochastic geometry, so studying the properties of even the simplest functionals of convex hulls, such as the number of vertices or the area, is not a simple task. This explains the fact that before the well-known work of P. Groeneboom [6] on the central limit theorem for the number of vertices of a convex hull, the main progress in this area was considered to be the study of asymptotic expressions for the average values of such functionals (see, for example, [4, 18, 17]. Due to the complexity of the object of study, research on asymptotic expressions of dispersion was not conducted until the publication of articles by C. Buchta [1, 2] and J. Pardon [14, 15].

Groeneboom's [6] main achievement is his use of the well-known property of homogeneous binomial point processes, which states that near the boundary of the support, such a process is almost indistinguishable from a homogeneous Poisson point process inside a

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"segment" of a parabola. This idea allowed him to reduce the number of problems to be solved when studying the asymptotic properties of binomial point processes. Following P. Groeneboom's work [6], we believe it is sufficient to study the properties of the vertex process of the convex hull generated by a inhomogeneous Poisson point process inside a parabola.

2. Statement of the problem and results

Denote the smallest root of the equation by b_n:

$$nx^{-(\beta+\frac{1}{2})}L(x) = 1, \tag{1}$$

where $L(x)$ is the slowly varying function (s.v.f) in sense of Karamata and $\beta \geq 1$. Below, in the definition of the intensity measure, differentiability of $L(x)$ is required, therefore from the integral representation s.v.f. (see E. Seneta [16]) we accept

$$L(u) = \exp\left\{\int_1^u \frac{\varepsilon(t)}{t}dt\right\}, \quad \varepsilon(t) \to 0, \quad t \to \infty. \tag{2}$$

From relation (1) according to [16], there exists some s.v.f. $L^*(x)$ such that $b_n = n^{\frac{2}{2\beta+1}}L^*(x)$. Therefore, $b_n \to \infty$ as $n \to \infty$.

Let

$$R_n = \left\{(x, y) : \frac{x^2}{2b_n} \leq y\right\} \subset \mathrm{R}^2.$$

We introduce the following measure

$$\Lambda_n(A) = \begin{cases} \dfrac{1}{2\pi L(b_n)\sqrt{b_n}} \displaystyle\iint\limits_A \frac{\partial}{\partial y}\left[\left(y - \frac{x^2}{2b_n}\right)^\beta L\left(\frac{b_n}{y - \frac{x^2}{2b_n}}\right)\right] dxdy & \text{if } A \subset R_n, \\ 0 \quad \text{if } A \not\subset R_n, \end{cases} \tag{3}$$

if $\beta = 1$ and $L(x) = 2 - \frac{1}{x}$ for $x \geq 1$, then the measure under consideration coincides with the measure of P.Groeneboom [6].

Let $\Pi_n(\cdot)$ be a inhomogeneous Poisson point process (i.p.p.p.) with intensity $\Lambda_n(\cdot)$, and let $(X_1, Y_1), (X_2, Y_2), \ldots$ be realizations of i.p.p.p. $\Pi_n(\cdot)$ is contracted measure to R_n (in what follows, denoted by $\Pi(R_n)$). Denote the convex hulls generated by these random points by C_n and their set of vertices by Z_n.

Let

$$e_0 = (0, 1) \tag{4}$$

Denote by $z_0 \in Z_n$ the vertices for which $(e_0, z - z_0) \geq 0$ for all $z \in Z_n$.

It is obvious that z_0 is determined uniquely and almost certainly.

In this case, the straight line

$$(e_0, z - z_0) = 0 \tag{5}$$

is the supporting line for C_n.

We are interested in the asymptotic properties of the domain bounded between perimeters C_n and parabola $y = \frac{x^2}{2b_n}$.

Consider the domain bounded by parabola $y = \frac{x^2}{2b_n}$ and the supporting line (5). Denote the set of interior points of this domain by δ_0 and the measures of this domain by

$$\xi_0 = \Lambda_n(\delta_0). \tag{6}$$

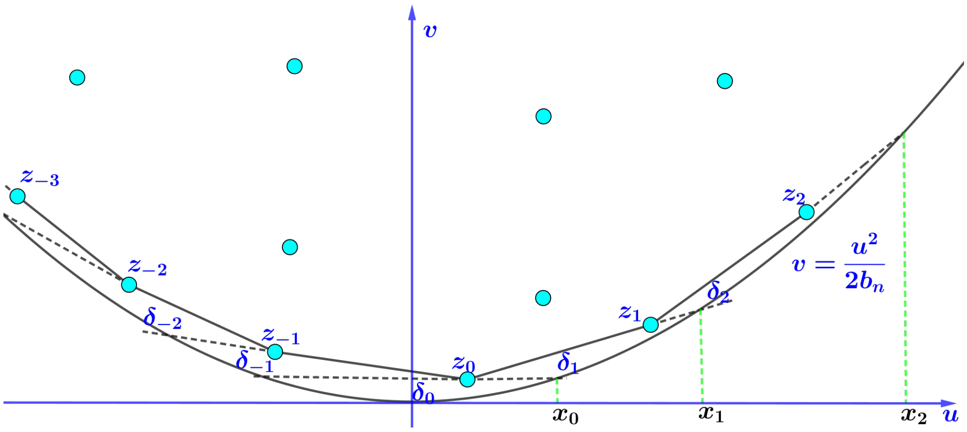

FIGURE 1. Illustration of z_j and δ_j.

Let $z_0 = (u_0, v_0)$, then it is easy to see that

$$\xi_0 = \Lambda_n(\delta_0) = \frac{1}{2\pi\sqrt{b_n}L(b_n)} \int_0^{v_0} dy \int_{-\sqrt{2b_n v_0}}^{\sqrt{2b_n v_0}} \frac{\partial}{\partial y} \left\{ \left(y - \frac{x^2}{2b_n} \right)^{\beta} L\left(\frac{b_n}{y - \frac{x^2}{2b_n}} \right) \right\} dx =$$

$$= \frac{\sqrt{2}}{\pi L(b_n)} \int_0^{v_0} dy \frac{\partial}{\partial y} \left\{ y^{\beta+\frac{1}{2}} \int_0^{\sqrt{2b_n v_0}} \left(1 - \frac{x^2}{2yb_n} \right)^{\beta} L\left(\frac{b_n}{y\left(1 - \frac{x^2}{2yb_n}\right)} \right) \right\} d\left(\frac{x}{\sqrt{2yb_n}} \right) =$$

$$= \frac{\sqrt{2}}{\pi L(b_n)} \int_0^{v_0} dy \frac{\partial}{\partial y} \left\{ y^{\beta+\frac{1}{2}} \int_0^1 (1-t^2)^{\beta} L\left(\frac{b_n}{y(1-t^2)} \right) dt \right\} =$$

$$= \frac{1}{\sqrt{2}\pi L(b_n)} \int_0^{v_0} dy \frac{\partial}{\partial y} \left\{ y^{\beta+\frac{1}{2}} \int_0^1 \frac{t^{\beta} L\left(\frac{b_n}{yt}\right)}{\sqrt{1-t}} dt \right\} = \frac{v_0^{\beta+\frac{1}{2}}}{\sqrt{2}\pi L(b_n)} \int_0^1 \frac{t^{\beta} L\left(\frac{b_n}{yt}\right)}{\sqrt{1-t}} dt$$

Let $\zeta_0 = \lambda(\delta_0)$, it is easy to see that

$$\zeta_0 = \int_{-x_0}^{x_0} \left(v_0 - \frac{u^2}{2b_n} \right) du = \int_{-\sqrt{2b_n v_0}}^{\sqrt{2b_n v_0}} \left(v_0 - \frac{u^2}{2b_n} \right) du = 2\int_0^{\sqrt{2b_n v_0}} \left(v_0 - \frac{u^2}{2b_n} \right) du =$$

$$= \frac{4\sqrt{2b_n} v_0^{\frac{3}{2}}}{3}, \tag{7}$$

where $\lambda(\cdot)$ is the Lebesgue measure and $x_0 = \sqrt{2b_n v_0}$.

Let

$$\eta_0 = \frac{u_0 + x_0}{2x_0} = \frac{u_0}{2\sqrt{2b_n \nu_0}} + \frac{1}{2}. \tag{8}$$

Then from (3) and (4) it is easy to obtain that

$$u_0 = x_0(2\eta_0 - 1) \quad \text{and} \quad b_n\zeta_0 = \frac{2x_0^3}{3}. \tag{9}$$

Number the vertices C_n, going around the boundary counterclockwise. Since z_0 is already defined, each of the vertices thereby receives its own number j, $-\infty < j < \infty$. Denote by x_j, $j \geq 1$ the abscissa of the intersection of the parabola $v = \frac{u^2}{2b_n}$ and the straight lines passing through vertices z_{j-1} and z_j respectively. Similarly, denote by $x_j, j \leq -1$ the abscissa of the intersection of the parabola $v = \frac{u^2}{2b_n}$ and the straight lines passing through z_j, z_{j+1}, respectively.

Let $\delta_j,\ j \neq 0$ be a set of interior points of a domain bounded by a segment connecting points z_{j-1} and $\left(x_{j-1}, x_{j-1}^2/(2b_n)\right),\ z_{j-1}$ and $\left(x_j, x_j^2/(2b_n)\right)$, with parabola $v = \frac{u^2}{2b_n}$ if $j \geq 1$, a segment connecting points z_{j+1} and $\left(x_{j+1}, x_{j+1}^2/(2b_n)\right),\ z_{j+1}$ and $\left(x_j, x_j^2/(2b_n)\right)$, with parabola $v = \frac{u^2}{2b_n}$ if $j \leq -1$. Let $\xi_j = \Lambda_n(\delta_j)$.

The following is true

Theorem 2.1. *If (1) – (5) are satisfied, then*

a) $P\left(z_0 \in (du_0, dv_0)\right)$

$$= \frac{1}{2\pi\sqrt{b_n}L(b_n)} \exp\left\{-\frac{v_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}L(b_n)} \int_0^1 \frac{t^\beta L\left(\frac{b_n}{\nu_0 t}\right)}{\sqrt{1-t}} dt\right\}$$

$$\cdot \frac{\partial}{\partial v_0}\left\{\left(v_0 - \frac{u_0^2}{2b_n}\right)^\beta L\left(\frac{b_n}{v_0 - \frac{u_0^2}{2b_n}}\right)\right\} du_0 dv_0;$$

b) $P\left(z_1 \in (du, dv)\,/z_0 = (u_0, v_0)\right)$

$$= \frac{1}{2\pi\sqrt{b_n}L(b_n)} \exp\left\{-\frac{1}{\sqrt{2\pi}L(b_n)} \int_{u_0-\rho_1 b_n}^{\sqrt{2b_n s_1}} \left(s_1 - \frac{t^2}{2b_n}\right)^\beta L\left(\frac{b_n}{s_1 - \frac{t^2}{2b_n}}\right) dt\right.$$

$$\left. - \int_{u_0}^{\sqrt{2b_n v_0}} \left(v_0 - \frac{t^2}{2b_n}\right)^\beta L\left(\frac{b_n}{v_0 - \frac{t^2}{2b_n}}\right) dt\right\} \cdot \frac{\partial}{\partial v}\left\{\left(v - \frac{u^2}{2b_n}\right)^\beta L\left(\frac{b_n}{v - \frac{u^2}{2b_n}}\right)\right\} du dv,$$

where $\rho_1 = \frac{v-v_0}{u-u_0}$ *and* $s_1 = v_0 - \rho_1 u_0 + \frac{\rho_1^2 b_n}{2}$.

c) $P\left(z_{i+1} \in (du_{i+1}, dv_{i+1})\,/z_{i-1} = (u_{i-1}, v_{i-1}),\ z_i = (u_i, v_i)\right)$

$$= \frac{1}{2\pi\sqrt{b_n}L(b_n)} \exp\left\{-\frac{1}{\sqrt{2\pi}L(b_n)}\left(s_i^{\beta+\frac{1}{2}}(b) \int_{u_i(b)}^1 (1-t^2)^\beta L\left(\frac{b_n}{s_i(b)(1-t^2)}\right) dt\right.\right.$$

$$\left.\left. - s_i^{\beta+\frac{1}{2}}(a) \int_{u_i(a)}^1 (1-t^2)^\beta L\left(\frac{b_n}{s_i(a)(1-t^2)}\right) dt\right)\right\}$$

$$\cdot \frac{\partial}{\partial v_{i+1}}\left\{\left(v_{i+1} - \frac{u_{i+1}^2}{2b_n}\right)^\beta L\left(\frac{b_n}{v_{i+1} - \frac{u_{i+1}^2}{2b_n}}\right)\right\} du_{i+1} dv_{i+1},$$

where $a = \frac{v_i - v_{i-1}}{u_i - u_{i-1}},\ b = \frac{v_{i+1} - v_i}{u_{i+1} - u_i},\ s_i(a) = v_i - a u_i + \frac{a^2 b_n}{2}$ *and* $u_i(a) = \frac{u_i - a b_n}{\sqrt{2b_n s_i(a)}}$.

Proof. The same procedure as in the proof of Lemma 3.1 [10] is conducted. Denote by $A_{\Delta v}^+(u_0, v_0)$ and $A_{\Delta v}^-(u_0, v_0)$ the sets bounded by lines $v = v_0 + \Delta v,\ v = \frac{u^2}{2b_n}$ and $v = v_0 - \Delta v,\ v = \frac{u^2}{2b_n}$, respectively.

Let $\Delta_{u,v} = [u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v]$ and let for the definiteness $\Delta u > 0, \Delta v > 0$ (see Fig. 2).

Denote by $\pi_n(A)$ the number of points of implementation of i.p.p.p. $\Pi(\cdot)$ in A, then from the definition of a Poisson point process we have

$$P(z_0 \in \Delta_{u,v}) \leq P\left(\pi_n(\Delta_{u,v}) \geq 1, \pi_n\left(A_{\Delta v}^-(u_0, v_0)\right) = 0\right).$$

From the independence of the increment of the Poisson process, we obtain an upper bound:

$$P(z_0 \in \Delta_{u,v}) \leq P\left(\pi_n(\Delta_{u,v}) \geq 1\right) \cdot P\left(\pi_n\left(A_{\Delta v}^-(u_0, v_0)\right) = 0\right) =$$

$$= \sum_{k=1}^{\infty} \frac{\left(\Lambda_n(\Delta_{u,v})\right)^k}{k!} \exp\left\{-\Lambda_n(\Delta_{u,v})\right\} \cdot \exp\left\{-\Lambda_n(A_{\Delta v}^-(u_0, v_0))\right\} \leq$$

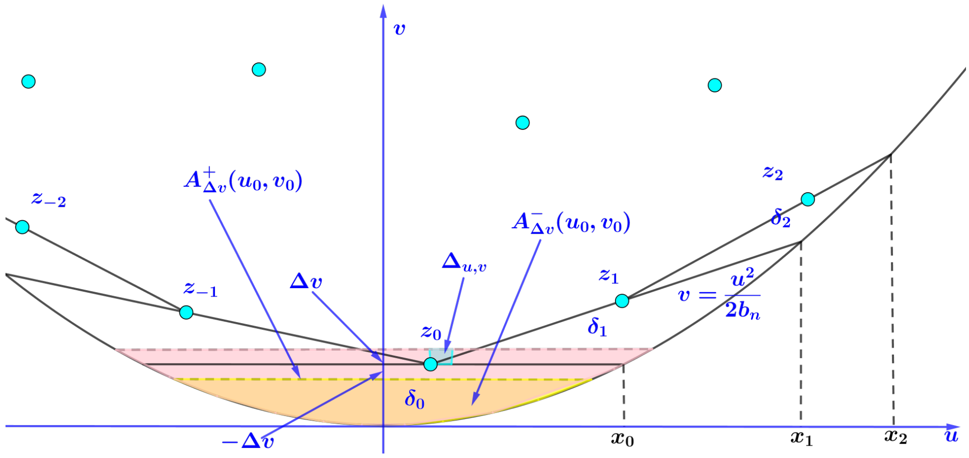

FIGURE 2. Illustration of $z_0, \Delta_{u,v}, A^-_{\Delta v}(u_0, v_0)$ and $A^+_{\Delta v}(u_0, v_0)$.

$$\leq \Lambda_n(\Delta_{u,v}) \cdot \exp\{-\Lambda_n(\Delta_{u,v})\} \cdot (1 + O(\Lambda_n(\Delta_{u,v}))) \cdot \exp\left\{-\Lambda_n(A^-_{\Delta v}(u_0, v_0))\right\}. \tag{10}$$

On the other side,

$$P(z_0 \in \Delta_{u,v}) \geq P\left(\pi_n(\Delta_{u,v}) = 1, \pi_n\left(A^+_{\Delta v}(u_0, v_0)\right) = 0\right). \tag{11}$$

Similar to (10), we obtain a lower bound:

$$P(z_0 \in \Delta_{u,v}) \geq P(\pi_n(\Delta_{u,v}) = 1) \cdot P\left(\pi_n\left(A^+_{\Delta v}(u_0, v_0)\right) = 0\right) \geq$$

$$\geq \Lambda_n(\Delta_{u,v}) \cdot \exp\{-\Lambda_n(\Delta_{u,v})\} \cdot \exp\left\{-\Lambda_n(A^+_{\Delta v}(u_0, v_0))\right\}. \tag{12}$$

By definition of measure $\Lambda_n(\cdot)$, for $\Delta u \to 0, \Delta v \to 0$ we have

$$\Lambda_n(\Delta_{u,v}) = \frac{\partial}{\partial v_0}\left(\left(v_0 - \frac{u_0^2}{2b_n}\right)^{\beta} L\left(\frac{b_n}{v_0 - \frac{u_0^2}{2b_n}}\right)\right) \Delta u \Delta v (1 + o(1)). \tag{13}$$

Thus, for $\Delta u \to 0, \Delta v \to 0$ we have

$$\Lambda_n(A^+_{\Delta v}(u_0, v_0)) = \Lambda_n(\delta_0)(1 + o(1)), \quad \Lambda_n(A^-_{\Delta v}(u_0, v_0)) = \Lambda_n(\delta_0)(1 + o(1)).$$

From this and from (10) - (13) follows the proof of the first assertion of the theorem. Now we will prove the third assertion from which the second one follows.

Reasoning in the same way as in the proof of the first assertion, it is easy to see that

$$P\left(z_{i+1} \in (du_{i+1}, dv_{i+1}) / z_{i-1} = (u_{i-1}, v_{i-1}), z_i = (u_i, v_i)\right)$$

$$= P\left(\overline{\pi}\left(\Lambda_n(\delta_{i+1})\right) = 0\right) \frac{\partial}{\partial v_{i+1}}\left(\left(v_{i+1} - \frac{u_{i+1}^2}{2b_n}\right)^{\beta} L\left(\frac{b_n}{v_{i+1} - \frac{u_{i+1}^2}{2b_n}}\right)\right) du_{i+1} dv_{i+1}$$

$$= \exp\{-\Lambda_n(\delta_{i+1})\} \frac{\partial}{\partial v_{i+1}}\left(\left(v_{i+1} - \frac{u_{i+1}^2}{2b_n}\right)^{\beta} L\left(\frac{b_n}{v_{i+1} - \frac{u_{i+1}^2}{2b_n}}\right)\right) du_{i+1} dv_{i+1}.$$

On the other hand, we have
$\Lambda_n(\delta_{i+1})$

$$= \frac{1}{2\pi\sqrt{b_n} L(b_n)} \left\{ \int_{u_i}^{bb_n + \sqrt{2b_n s_i(b)}} \left(b(\tau - u_i) + v_i - \frac{\tau^2}{2b_n}\right)^{\beta} L\left(\frac{b_n}{b(\tau - u_i) + v_i - \frac{\tau^2}{2b_n}}\right) d\tau\right.$$

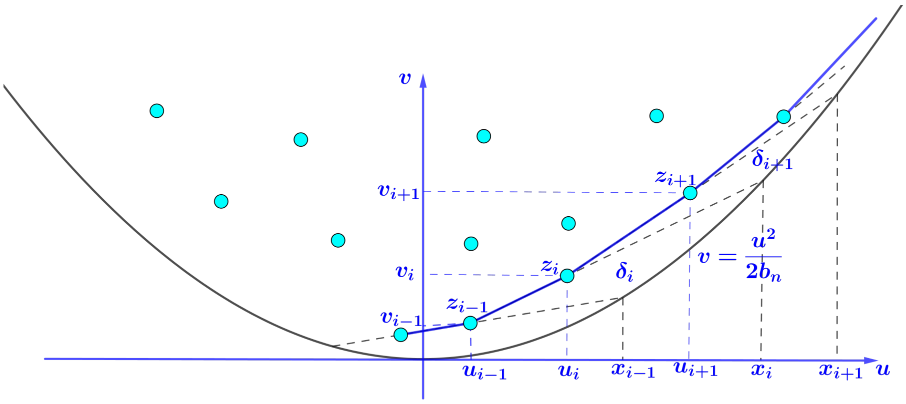

FIGURE 3. Illustrations of z_{i-1}, z_i, z_{i+1}.

$$- \int_{u_i}^{ab_n+\sqrt{2b_n s_i(a)}} \left(a(\tau - u_i) + v_i - \frac{\tau^2}{2b_n} \right)^{\beta} L \left(\frac{b_n}{a(\tau - u_i) + v_i - \frac{\tau^2}{2b_n}} \right) d\tau \Bigg\}$$

$$= \frac{1}{2\pi\sqrt{b_n} L(b_n)} \left\{ \int_{u_i}^{bb_n+\sqrt{2b_n s_i(b)}} \left(s_i(b) - \frac{(\tau - bb_n)^2}{2b_n} \right)^{\beta} L \left(\frac{b_n}{s_i(b) - \frac{(\tau-bb_n)^2}{2b_n}} \right) d\tau \right.$$

$$\left. - \int_{u_i}^{ab_n+\sqrt{2b_n s_i(a)}} \left(s_i(a) - \frac{(\tau - ab_n)^2}{2b_n} \right)^{\beta} L \left(\frac{b_n}{s_i(a) - \frac{(\tau-ab_n)^2}{2b_n}} \right) d\tau \right\}.$$

In the last expression, replacing variable $\frac{\tau - bb_n}{\sqrt{2b_n s_i(b)}}$ through t in the first interval and again replacing $\frac{\tau - ab_n}{\sqrt{2b_n s_i(a)}}$ through t in the second interval, we obtain a proof of the third assertion of the theorem.

To prove the second assertion of Theorem 2.1 instead of

$$P\left(z_{i+1} \in (du_{i+1}, dv_{i+1}) / z_{i-1} = (u_{i-1}, v_{i-1}), z_i = (u_i, v_i)\right)$$

we write

$$P\left(z_1 \in (du_1, dv_1) / z_0 = (u_0, v_0)\right),$$

then from the last expression, in the case $i = 0$, $a = 0$ presenting considerations similar to the ones given above, we obtain assertion c) of Theorem 2.1. □

Then, we consider convex hulls generated by homogeneous Poisson point processes.

3. PROPERTIES OF THE VERTEX OF A CONVEX HULL GENERATED BY THE HOMOGENEOUS POISSON POINT PROCESS

An interesting property of homogeneous Poisson point processes (h.p.p.p.) was established when using methods to prove Theorem 2.1. A similar assumption was first considered in [12], and then in [5], [7], [11], [13].

Let now $\Pi(\cdot)$ be a h.p.p.p. restricted in R_n, that is, $\lambda(\cdot)$ is a measure of intensity of the Poisson law – Lebesgue measure. Next, let $(X_1, Y_1); (X_2, Y_2); \ldots$ be the implementation of h.p.p.p. in R_n, C_n be the convex hulls generated by these random points and Z_n – their sets of vertices.

In a similar way, the elements of sets Z_n are denoted by $z_j, -\infty < j < \infty$.

Note that

$$\eta_0 = \frac{u_0 + x_0}{2x_0} = \frac{u_0}{2\sqrt{2b_n v_0}} + \frac{1}{2} \quad \text{and} \quad \zeta_0 = \frac{4\sqrt{2b_n} v_0^{\frac{3}{2}}}{3}.$$

Then the following theorem is true.

Theorem 3.1. *If conditions (1) – (5) are satisfied, then ζ_0 and η_0 are independent random variables, moreover, ζ_0 is the standard exponentially distributed variable, and η_0 is the uniformly distributed variable.*

Proof. If $\Lambda_n(\cdot)$ is replaced with $\lambda(\cdot)$, then from (6), it follows that $\xi_0 = \Lambda_n(\delta_0) = \zeta_0$, then from assertion a) of Theorem 2.1 the joint density of ζ_0 and η_0 is $e^{-\zeta_0}$. Hence, similarly to the proof of Lemma 3.1 [10] from (3–4) and (6), it suffices to show that after the change of variables given below, the Jacobian equals to 1.

$$\begin{cases} \zeta_0 = \dfrac{4\sqrt{2b_n} v_0^{\frac{3}{2}}}{3}, \\ \eta_0 = \dfrac{u_0}{2\sqrt{2b_n v_0}} + \dfrac{1}{2}, \end{cases}$$

where $\zeta_0 \geq 0,\ 0 \leq \eta_0 \leq 1$. Hence, we obtain:

$$\frac{\partial \zeta_0}{\partial u_0} = 0, \ \frac{\partial \zeta_0}{\partial v_0} = 2\sqrt{2b_n v_0}, \ \frac{\partial \eta_0}{\partial u_0} = \frac{1}{2\sqrt{2b_n v_0}}, \ \frac{\partial \eta_0}{\partial v_0} = -\frac{u_0}{4v_0\sqrt{2b_n v_0}}.$$

Then the Jacobian transformation is $|J| = 1$. The proof of the Theorem is completed. □

Then let $\delta_j, -\infty < j < \infty$, have the same meaning as in the previous section and $\zeta_j = \lambda(\delta_j)$. To simplify, consider the case of $j > 0$; the case of $j < 0$ is treated in a similar way. Then it is easy to verify that

$$\begin{aligned} \zeta_{j+1} &= \int_{u_j}^{x_{j+1}} \left(\rho_{j+1}(t - u_j) + v_j - \frac{t^2}{2b_n} \right) dt - \int_{u_j}^{x_j} \left(\rho_j(t - u_j) + v_j - \frac{t^2}{2b_n} \right) dt \\ &= \rho_{j+1} \frac{(x_{j+1} - u_j)^2}{2} + v_j(x_{j+1} - u_j) - \frac{1}{6b_n}(x_{j+1}^3 - u_j^3) \\ &\quad - \left(\rho_j \frac{(x_j - u_j)^2}{2} + v_j(x_j - u_j) - \frac{1}{6b_n}(x_j^3 - u_j^3) \right), \end{aligned} \tag{14}$$

where $\rho_j = \frac{v_j - v_{j-1}}{u_j - u_{j-1}}$.

Let

$$\eta_{j+1} = \left(\frac{u_{j+1} - u_j}{x_{j+1} - u_j} \right)^2, \ j \geq 0 \tag{15}$$

From here it is easy to see that

$$\zeta_{j+1} \geq 0, \ 0 \leq \eta_{j+1} \leq 1.$$

Theorem 3.2. *If conditions (1) – (5) are satisfied, then $\zeta_j,\ \eta_k,\ j, k \geq 1$ are independent random variables, and*

$$\zeta_j \overset{dis}{=} \zeta_0, \quad \eta_k \overset{dis}{=} \eta_0.$$

Proof. Similar to the proof of Theorem 3.1, if $\Lambda_n(\cdot)$ is replaced with $\lambda(\cdot)$, then $\xi_i = \Lambda_n(\delta_i) = \zeta_i$. From this and from assertions c) and c) of Theorem 2.1 for all $i \geq 1$, the joint conventional density ζ_i and η_i, under conditions ζ_{i-1} and η_{i-1} is e^{ζ_i}. Therefore, to prove Theorem 3.2, it suffices to show that the Jacobian of the change of variables

$$\begin{cases} \zeta_{j+1} = \rho_{j+1}\dfrac{(x_{j+1}-u_j)^2}{2} + v_j(x_{j+1}-u_j) - \dfrac{1}{6b_n}(x_{j+1}^3 - u_j^3) - \\ \qquad - \left(\rho_j \frac{(x_j-u_j)^2}{2} + v_j(x_j-u_j) - \frac{1}{6b_n}(x_j^3-u_j^3)\right), \\ \eta_{j+1} = \left(\dfrac{u_{j+1}-u_j}{x_{j+1}-u_j}\right)^2, \end{cases}$$

$j \geq 0$. Here, it is easy to verify that $\zeta_{j+1} \geq 0,\ \ 0 \leq \eta_{j+1} \leq 1$.

On the other hand, we have

$$\frac{\partial \rho_{j+1}}{\partial u_{j+1}} = -\frac{v_{j+1}-v_j}{(u_{j+1}-u_j)^2}, \quad \frac{\partial \rho_{j+1}}{\partial v_{j+1}} = \frac{1}{u_{j+1}-u_j}.$$

Making a change of variables in the form (14), (15) and taking into account that x_{j+1} is the solution to the following equation

$$\rho_{j+1}(x-u_j) + v_j = \frac{x^2}{2b_n} \tag{16}$$

we obtain

$$\frac{\partial \zeta_{j+1}}{\partial u_{j+1}} = -\frac{v_{j+1}-v_j}{(u_{j+1}-u_j)^2} \cdot \frac{(x_{j+1}-u_j)^2}{2} + \left(\rho_{j+1}(x_{j+1}-u_j) + v_j - \frac{x_{j+1}^2}{2b_n}\right) \cdot \frac{\partial x_{j+1}}{\partial u_{j+1}}$$
$$= -\frac{v_{j+1}-v_j}{(u_{j+1}-u_j)^2} \cdot \frac{(x_{j+1}-u_j)^2}{2},$$
$$\frac{\partial \zeta_{j+1}}{\partial v_{j+1}} = \frac{1}{u_{j+1}-u_j} \cdot \frac{(x_{j+1}-u_j)^2}{2} + \left(\rho_{j+1}(x_{j+1}-u_j) + v_j - \frac{x_{j+1}^2}{2b_n}\right) \cdot \frac{\partial x_{j+1}}{\partial v_{j+1}}$$
$$= \frac{(x_{j+1}-u_j)^2}{2(u_{j+1}-u_j)},$$
$$\frac{\partial \eta_{j+1}}{\partial u_{j+1}} = \frac{2(u_{j+1}-u_j)}{x_{j+1}-u_j} \cdot \frac{1}{x_{j+1}-u_j} - \frac{2(u_{j+1}-u_j)^2}{(x_{j+1}-u_j)^3} \cdot \frac{\partial x_{j+1}}{\partial u_{j+1}},$$
$$\frac{\partial \eta_{j+1}}{\partial v_{j+1}} = -\frac{2(u_{j+1}-u_j)^2}{(x_{j+1}-u_j)^3} \cdot \frac{\partial x_{j+1}}{\partial v_{j+1}},$$

From here follows the Jacobians transformation

$$J = \frac{\partial \zeta_{j+1}}{\partial u_{j+1}} \cdot \frac{\partial \eta_{j+1}}{\partial v_{j+1}} - \frac{\partial \zeta_{j+1}}{\partial v_{j+1}} \cdot \frac{\partial \eta_{j+1}}{\partial u_{j+1}} = -1 + \frac{v_{j+1}-v_j}{x_{j+1}-u_j} \cdot \frac{\partial x_{j+1}}{\partial v_{j+1}} + \frac{u_{j+1}-u_j}{x_{j+1}-u_j} \cdot \frac{\partial x_{j+1}}{\partial u_{j+1}} \tag{17}$$

It is easy to verify that the solution to the equation has the following form:

$$x_{j+1} = \rho_{j+1}b_n + \sqrt{2b_n\left(v_j - \rho_{j+1}u_j + \frac{\rho_{j+1}^2 b_n}{2}\right)}$$

then

$$\frac{\partial x_{j+1}}{\partial u_{j+1}} = \frac{\partial x_{j+1}}{\partial \rho_{j+1}} \cdot \frac{\partial \rho_{j+1}}{\partial u_{j+1}} = -b_n \cdot \frac{x_{j+1}-u_j}{\sqrt{2b_n s_{j+1}}} \cdot \frac{v_{j+1}-v_j}{(u_{j+1}-u_j)^2},$$
$$\frac{\partial x_{j+1}}{\partial v_{j+1}} = \frac{\partial x_{j+1}}{\partial \rho_{j+1}} \cdot \frac{\partial \rho_{j+1}}{\partial v_{j+1}} = b_n \cdot \frac{x_{j+1}-u_j}{\sqrt{2b_n s_{j+1}}} \cdot \frac{1}{u_{j+1}-u_j},$$

where $s_{j+1} = v_j - \rho_{j+1}u_j + \frac{\rho_{j+1}^2 b_n}{2}$. From this and from (16), (15) we obtain $|J| = 1$. The theorem is proven.

□

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