Theory of Stochastic Processes
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B. I. KOPYTKO AND M. I. PORTENKO

SIX LESSONS ON THE THEORY OF DIFFUSION PROCESSES

The itinerary from the notion of a diffusion process to that of a generalized diffusion process is split into six lessons. Numerous exercises throughout its extent make this minicourse look like a collection of etudes for those ones who are interested in the theory of diffusion processes.

Introduction.

In response to the request of editorial board of this journal, we present our minicourse on the theory of diffusion processes consisting of six lessons. Being quite conscious of impossibility to squeeze any considerable part of that theory into the framework of so few lessons, we still venture on publishing some synopsis of the minicourse with the purpose of showing how does one natural modification in understanding the concept of a diffusion process result in an essential extension of the original set of diffusion processes. That extended set turns out to contain some processes that can be treated as diffusion ones only in a certain generalized sense. Moreover, generally speaking, such a generalized diffusion process cannot serve as a mathematical model for describing any dynamical system evolving under the influence of random factors: some new kind of interpretations should be proposed.

To clarify our idea in more details, remind that according to Kolmogorov's definition*), a diffusion process in a d-dimensional Euclidean space $\mathbb{R}^d$ is determined by its local characteristics, that is, by two functions defined at any instant of time and any point of $\mathbb{R}^d$: one of them is $\mathbb{R}^d$-valued and is called drift vector; the other one called diffusion operator takes on its values from the set of all linear operators in $\mathbb{R}^d$ being non-negative definite. Denote these functions, respectively, by $a(t,x)$ and $b(t,x)$ for $t\geq 0$ and $x\in\mathbb{R}^d$.

The following result was proved by A.N.Kolmogorov almost 100 years ago (see Lesson 2 below). Let $P(s,x,t,dy)$ for $0\leq s<t\leq T$ and $x\in\mathbb{R}^d$ be transition probability of a diffusion process in $\mathbb{R}^d$ with its local characteristics given by continuous functions $(a(t,x))_{(t,x)\in\mathcal{D}_T}$ and $(b(t,x))_{(t,x)\in\mathcal{D}_T}$, where the notation $\mathcal{D}_T=\{(t,x): t\in[0,T], x\in\mathbb{R}^d\}$ is used for $T>0$. Suppose that a real-valued continuous bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$ is given such that the function

$$u(s,x)=\int_{\mathbb{R}^d}\varphi(y)P(s,x,t,dy),\ s\in[0,t),\ x\in\mathbb{R}^d, \tag{1}$$

for fixed $t\in(0,T]$ is twice continuously differentiable in the argument x. Then this function is differentiable in s as well and it satisfies the equation

$$u'_s(s,x)+(a(s,x),u'_x(s,x))+\frac{1}{2}\operatorname{Tr}(b(s,x)u''_{xx}(s,x))=0 \tag{2}$$

in the domain $(s,x)\in[0,t)\times\mathbb{R}^d$; by the same token the final (in opposite to initial) condition

$$u(t-,x)=\varphi(x),\ x\in\mathbb{R}^d, \tag{3}$$

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*) At the end of 1920s, A.N.Kolmogorov singled out a class of Markov processes that somewhat later came to be known as diffusion ones.

is fulfilled.

Equation (2) is called the Kolmogorov backward equation. One of the consequences of it is the following interpretation of a diffusion process.

If it so happens that $b(t,x) \equiv 0$, then the corresponding diffusion process is nothing else but a dynamical system (maybe, not unique) that is generated by the vector field $(a(t,x))_{(t,x)\in\mathcal{D}_T}$. In the general cases ($b(t,x) \not\equiv 0$), our process can be treated as the result of perturbing the dynamical system mentioned above by some random factors generated by the operator field $(b(t,x))_{(t,x)\in\mathcal{D}_T}$.

The development of methods for constructing a process from given $\mathbb{R}^d$-valued function $(a(t,x))_{(t,x)\in\mathcal{D}_T}$ and operator-valued function $(b(t,x))_{(t,x)\in\mathcal{D}_T}$ is one of the most important problems in the theory of diffusion processes. Some classical results of the kind are formulated in Lesson 3. The so-called parametrix method for constructing the fundamental solution of equation (2) together with the maximum principle for that equation allow one to obtain transition probability density of the diffusion process desired. This construction can be fulfilled under the following assumptions on the given functions a and b : they are bounded and smooth enough and besides, the function b is supposed to be uniformly nonsingular (see Lesson 3). Suppose that given functions: $\mathbb{R}^d$-valued $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ and operator-valued $(b(s,x))_{(s,x)\in\mathcal{D}_T}$ satisfy these conditions and let $g(s,x,t,y), 0 \le s < t \le T, x \in \mathbb{R}^d, y \in \mathbb{R}^d$, be transition probability density of the diffusion process in $\mathbb{R}^d$ whose local characteristics are given by those functions. In other words, g is the fundamental solution of equation (2). Denote by $g_0(s,x,t,y)$ for $0 \le s < t \le T, x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$ the fundamental solution of the equation

$$v'_s(s,x) + \frac{1}{2}\operatorname{Tr}(b(s,x)v''_{xx}(s,x)) = 0. \tag{4}$$

The assertions presented in Lesson 3 allow one to arrive at the following relations between the functions g and g_0

$$\begin{aligned} g(s,x,t,y) &= g_0(s,x,t,y) + \int_s^t d\tau \int_{\mathbb{R}^d} g_0(s,x,\tau,z)(a(\tau,z),\nabla_z g(\tau,z,t,y))dz, \\ g(s,x,t,y) &= g_0(s,x,t,y) + \int_s^t d\tau \int_{\mathbb{R}^d} g(s,x,\tau,z)(a(\tau,z),\nabla_z g_0(\tau,z,t,y))dz \end{aligned} \tag{5}$$

valid for $0 \le s < t \le T$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$.

These relations are known in mathematics as perturbation formulae. They inspire one more point of view on the diffusion process whose local characteristics are given by the functions a and b. Such a process is the result of perturbing the diffusion process with the local characteristics a_0 and b (the function a_0 is defined by the identity $a_0(t,x) \equiv 0$) by the vector field $(a(t,x))_{(t,x)\in\mathcal{D}_T}$.

We can now formulate the modification in understanding the notion of a diffusion process that was mentioned in the first paragraph of this introduction. In Kolmogorov's definition, the local characteristics of a diffusion process are determined as the *pointwise* limits as $\Delta s \downarrow 0$, of the following expressions ("pointwise" means the existence of the limits for any $s \in [0,T)$ and $x \in \mathbb{R}^d$)

$$\frac{1}{\Delta s}\int_{B_\varepsilon(x)} (y-x)P(s,x,s+\Delta s,dy) \text{ and } \frac{1}{\Delta s}\int_{B_\varepsilon(x)} (y-x,\theta)^2 P(s,x,s+\Delta s,dy),$$

where $\theta \in \mathbb{R}^d,\ B_\varepsilon(x) = \{y \in \mathbb{R}^d : |y-x| < \varepsilon\}$ for $x \in \mathbb{R}^d$ and $\varepsilon > 0$ (θ and ε are fixed) and $P(s,x,t,dy)$ for $0 \le s < t \le T,\ x \in \mathbb{R}^d$ is the designation for transition probability of the process under considerations; the first limit determines the drift vector $a(s,x)$ and the second one determines the form $(b(s,x)\theta,\theta)$.

In our modified definition, those limits are supposed to exist in the following sense: the limits

$$\lim_{\Delta s\downarrow 0}\int_0^T\int_{\mathbb{R}^d}\varphi(s,x)\Big[\frac{1}{\Delta s}\int_{B_\varepsilon(x)}(y-x)P(s,x,s+\Delta s,dy)\Big]dsdx, \tag{6}$$

$$\lim_{\Delta s\downarrow 0}\int_0^T\int_{\mathbb{R}^d}\varphi(s,x)\Big[\frac{1}{\Delta s}\int_{B_\varepsilon(x)}(y-x,\theta)^2P(s,x,s+\Delta s,dy)\Big]dsdx, \tag{7}$$

exist for any continuous compactly supported in $(0,T)\times\mathbb{R}^d$ function φ ($\theta\in\mathbb{R}^d$ and $\varepsilon>0$ are fixed). In other words, the existence of local characteristics of a Markov process is now supposed in a *weak* sense. It is natural to call generalized diffusion a Markov process in $\mathbb{R}^d$ whose local characteristics exist in this weak sense.

In Lessons 4–6, we show how to construct a generalized diffusion process in $\mathbb{R}^d$ such that its diffusion operator is given by a regular function $(b(t,x))_{(t,x)\in\mathcal{D}_T}$ and its drift vector is given by a function $(a(t,x))_{(t,x)\in\mathcal{D}_T}$ from the class L_p for some p being large enough (Lesson 4) or (Lessons 5–6) from the class of generalized functions of the form $(N(x)\delta_S(x))_{x\in\mathbb{R}^d}$, where S is a given hypersurface in $\mathbb{R}^d$, $(N(x))_{x\in S}$ is a given vector field and $(\delta_S(x))_{x\in\mathbb{R}^d}$ is a generalized function on $\mathbb{R}^d$ acting on a test function $(\psi(x))_{x\in\mathbb{R}^d}$ according to the rule $\langle\delta_S,\psi\rangle=\int_S\psi(x)d\sigma$ (in fact, some "generalization" of this generalized function is needed). The main devices for constructing processes of the kind are perturbation formulae (5).

As the reader can see, there is no dynamical system in $\mathbb{R}^d$ generated by the vector field $(a(t,x))_{(t,x)\in\mathcal{D}_T}$ considered in Lessons 5–6. We propose to interpret the corresponding generalized diffusion process as a diffusion one in a medium where some *membranes* are located on given hypersurfaces. It is a very interesting problem to investigate the behaviour of such a diffusion process near the membrane. Some results of the kind can be found in our recent publication (see [11]).

If we now add to what have been said before, that Lesson 1 contains some kind of concise introduction to the theory of Markov processes, then the reader must be able to realize the contents of our minicourse on the whole.

The exposition of the underlying ideas in the theory of diffusion processes was our goal in preparing this minicourse for publishing and "from diffusion to generalized diffusion" was our motto. Many details are hidden in exercises proposed for reader's thinking them over. In fact, several steps in proving the main assertions of Lessons 4–6 are presented as exercises provided with some hints. We are sure, those readers are able to cope with the exercises who have mastered the technique of the theory of heat potentials or even better, the technique of the parametrix method for constructing a fundamental solution to equation (2). Summarizing and resorting to terms of music, we can say that our minicourse is a collection of *etudes* for those readers who are interested in the theory of diffusion processes.

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Lesson 1. Markov processes.

1.1. Definition. Let the following objects be given:

- A measurable space $(\Omega,\mathcal{F})$; any point $\omega\in\Omega$ is interpreted as an elementary event; $\mathcal{F}$ is some σ-algebra of subsets of Ω, any $A\in\mathcal{F}$ is called an event.
- One more measurable space $(X,\mathcal{B})$ that is interpreted as the phase space; it is assumed that any single-point set is measurable, that is $\{x\}\in\mathcal{B}$ for all $x\in X$.

- A two-parametric family $(\mathcal{M}_t^s)_{s\le t}$ of σ-algebras of events such that $\mathcal{M}_\tau^\sigma \subseteq \mathcal{M}_t^s$ if $0 \le s \le \sigma \le \tau \le t$; for fixed $s \ge 0$, the minimal σ-algebra of events containing all the σ-algebras $\mathcal{M}_t^s$ for $t \ge s$ will be denoted by $\mathcal{M}^s$.
- An X-valued function $(x(t,\omega))_{t\ge 0,\omega\in\Omega}$ possessing the property: $\{\omega \in \Omega : x(t,\omega) \in \Gamma\}$ is an event from $\mathcal{M}_t^t$ for all $t \ge 0$ and $\Gamma \in \mathcal{B}$; as a rule, the second argument of this function will be omitted and the event above will be written briefly $\{x(t) \in \Gamma\}$; the minimal σ-algebra of events containing all the events of the kind $\{x(r) \in \Gamma\}, r \in [s,t], \Gamma \in \mathcal{B}$ is denoted by $\mathcal{N}_t^s$; it is clear that $\mathcal{N}_t^s \subseteq \mathcal{M}_t^s$ for all $0 \le s \le t$; the notation $\mathcal{N}^s$ for $s \ge 0$ will be used for the minimal σ-algebra of events containing $\mathcal{N}_t^s$ for all $t \in [s,+\infty)$.
- For any pair of $s \ge 0$ and $x \in X$, a probability measure $\mathbb{P}_{s,x}$ on the σ-algebra $\mathcal{M}^s$.

Suppose that the following conditions are fulfilled:

(i) $\mathbb{P}_{s,x}(\{x(s) = x\}) = 1$ for all $s \ge 0$ and $x \in X$;

(ii) for fixed $s \ge 0,\ t \ge s$ and $\Gamma \in \mathcal{B}$, the function $(\mathbb{P}_{s,x}(\{x(t) \in \Gamma\}))_{x\in X}$ is $\mathcal{B}$-measurable;

(iii) the equality $\mathbb{P}_{s,x}(\{x(t) \in \Gamma\}/\mathcal{M}_\tau^s) = \mathbb{P}_{\tau,x(\tau)}(\{x(t) \in \Gamma\})$ holds true $\mathbb{P}_{s,x}$-almost surely for all $0 \le s < \tau < t,\ x \in X$ and $\Gamma \in \mathcal{B}$.

Then we say that a Markov process is given and denote it by $(x(t), \mathcal{M}_t^s, \mathbb{P}_{s,x})$.

Sometimes we will say briefly: a Markov process $(x(t))_{t\ge 0}$ is given in the phase space $(X,\mathcal{B})$. The function in (ii) will be denoted by $P(s,x,t,\Gamma)$ for $0 \le s \le t, x \in X$ and $\Gamma \in \mathcal{B}$, that is $\mathbb{P}_{s,x}(\{x(t) \in \Gamma\}) = P(s,x,t,\Gamma)$; it is called transition probability of the Markov process $(x(t), \mathcal{M}_t^s, \mathbb{P}_{s,x})$. As a function of the fourth argument it is a probability measure on $(X,\mathcal{B})$. Moreover, the property (iii) implies the following equality

$$P(s,x,t,\Gamma) = \int_X P(\tau,z,t,\Gamma)P(s,x,\tau,dz) \tag{1}$$

valid for all $0 \le s < \tau < t,\ x \in X$ and $\Gamma \in \mathcal{B}$. This equality is called the Kolmogorov–Chapman equation.

The values of the measure $\mathbb{P}_{s,x}$ on $\mathcal{N}^s$ are completely determined by transition probability of the process. It is true for events of the kind $\bigcap_{k=1}^n \{x(t_k) \in \Gamma_k\}$ with an integer $n \ge 1$, instants of time $s < t_1 < \ldots < t_n$ and sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ from $\mathcal{B}$, since by induction on n, one can easily arrive at the equality

$$\mathbb{P}_{s,x}(\bigcap_{k=1}^n \{x(t_k) \in \Gamma_k\}) =$$

$$= \int_{\Gamma_1} P(s,x,t_1,dy_1) \int_{\Gamma_2} P(t_1,y_1,t_2,dy_2) \ldots \int_{\Gamma_n} P(t_{n-1},y_{n-1},t_n,dy_n).$$

Now, it is an easy exercise to verify that the equality

$$\mathbb{P}_{s,x}(A/\mathcal{M}_\tau^s) = \mathbb{P}_{\tau,x(\tau)}(A)$$

is held true $\mathbb{P}_{s,x}$-almost surely for all $0 \le s < \tau$ and $A \in \mathcal{N}^\tau$. Let $\mathbb{E}_{s,x}$ be the symbol for the expectation operator with respect to $\mathbb{P}_{s,x}$. Then the previous equality implies the following one $\mathbb{E}_{s,x}(\xi/\mathcal{M}_\tau^s) = \mathbb{E}_{\tau,x(\tau)}(\xi)$ valid $\mathbb{P}_{s,x}$-almost surely for all bounded $\mathcal{N}^\tau$-measurable random variables ξ. If additionally an $\mathcal{M}_\tau^s$-measurable bounded random variable η is given, we can assert that the equality

$$\mathbb{E}_{s,x}(\xi\eta) = \mathbb{E}_{s,x}(\eta\mathbb{E}_{\tau,x(\tau)}(\xi)) \tag{2}$$

holds true. This formula is useful.

As we have just seen, the measure $\mathbb{P}_{s,x}$ on $\mathcal{N}^s$ is determined by transition probability of the process. Suppose now that we have managed to solve the Kolmogorov–Chapman equation written for a given measurable space $(X,\mathcal{B})$ and ask ourselves whether there exists a Markov process in that space such that its transition probability coincides with that solution. The answer is positive if X is a complete separable metric space and $\mathcal{B}$ is the σ-algebra of all Borel measurable subsets of X. The proof of this statement is based on the well-known Kolmogorov theorem on consistent finite-dimensional distributions.

1.2. Continuity conditions. Two Markov processes in the same phase space (given, maybe, on different spaces of elementary events) are called stochastically equivalent if they have the same transition probability. So, any solution to the Kolmogorov–Chapman equation generates (if any) a class of stochastically equivalent Markov processes. It is natural to look for such a process in that class whose trajectories (that is, the functions $x(\cdot,\omega)$ for $\omega \in \Omega$) share as nice properties as possible. In the following statement, some condition imposed on transition probability of a Markov process in a complete metric space turns out to provide the existence of a stochastically equivalent process whose trajectories are continuous functions.

For a metric space X, we denote by $B_r(x)$ ($r > 0$ and $x \in X$) an open ball in X of radius r and its center located at x; $B_r(x)^c$ means the complement of $B_r(x)$.

Theorem. *Any Markov process in a complete metric space $(X, \mathcal{B})$ whose transition probability satisfies the condition*

$$\sup_{0\le s<t\le s+h\le T} \sup_{x\in X} P(s,x,t,B_\varepsilon(x)^c) = o(h), \text{ as } h \downarrow 0,$$

for all fixed $\varepsilon > 0$ and $T > 0$ is stochastically equivalent to a Markov process with its trajectories being continuous functions.

A Markov process in a metric space $(X, \mathcal{B})$ with continuous trajectories will be called continuous. For such a process it is natural to choose the space $\mathbb{C}([0,+\infty), X)$ of all X-valued continuous functions defined on $[0,+\infty)$ as the space of elementary events, so that $\omega = (\omega(s))_{s\ge 0}$. The function $(x(t,\omega))_{t\ge 0,\omega\in\Omega}$ is defined by $x(t,\omega) = \omega(t)$; the σ-algebra $\mathcal{M}_t^s$ for $0 \le s \le t < \infty$ coincides with the minimal σ-algebra of events containing any set of the kind $\{\omega(\cdot) : \ \omega(r) \in \Gamma\}$ with $r \in [s,t]$ and $\Gamma \in \mathcal{B}$; for $s \ge 0$, we have $\mathcal{M}^s = \bigvee_{t\ge s} \mathcal{M}_t^s$; for $s \ge 0$ and $x \in X$, the measure $\mathbb{P}_{s,x}$ on $\mathcal{M}^s$ is induced by the corresponding one of a given continuous Markov process in $(X, \mathcal{B})$.

1.3. Homogeneous Markov processes. A Markov process $(x(t), \mathcal{M}_t^s, \mathbb{P}_{s,x})$ in a phase space $(X, \mathcal{B})$ is called homogeneous if its transition probability $P(s,x,t,\Gamma)$, $s \le t, x \in X$ and $\Gamma \in \mathcal{B}$, possesses the following property: for all $t \ge 0, x \in X$ and $\Gamma \in \mathcal{B}$, the function $(P(s,x,s+t,\Gamma))_{s\ge 0}$ does not depend on s. If it is so, we put $P(t,x,\Gamma) = P(s,x,s+t,\Gamma)$ for $t \ge 0$, $x \in X$ and $\Gamma \in \mathcal{B}$. This function satisfies the following conditions ($\mathbb{1}_\Gamma(x)$ for $x \in X$ and $\Gamma \in \mathcal{B}$ is the notation for an indicator function):

- $P(0,x,\Gamma) = \mathbb{1}_\Gamma(x)$ for $x \in X$ and $\Gamma \in \mathcal{B}$;
- for fixed $t \ge 0$ and $\Gamma \in \mathcal{B}$, the function $(P(t,x,\Gamma))_{x\in X}$ is $\mathcal{B}$-measurable;
- for fixed $t \ge 0$ and $x \in X$, the function $(P(t,x,\Gamma))_{\Gamma\in\mathcal{B}}$ is a probability measure on $(X, \mathcal{B})$;
- for fixed $s > 0, t > 0, x \in X$ and $\Gamma \in \mathcal{B}$, the equality

$$P(s+t,x,\Gamma) = \int_X P(s,x,dy)P(t,y,\Gamma) \tag{1^0}$$

  holds true (this is a homogeneous version of the Kolmogorov–Chapman equation).

For a homogeneous process, there is no sense in fixing any initial instant of time in the measure $\mathbb{P}_{s,x}$: by a shift, it can always be chosen being equal to 0. For example, for $0 \le s < t$, $x \in X$ and $\Gamma \in \mathcal{B}$, we have

$$\mathbb{P}_{s,x}(\{x(t) \in \Gamma\}) = \mathbb{P}_{0,x}(\{x(t-s) \in \Gamma\}) = P(t-s,x,\Gamma).$$

If $0 \le s < t_1 < t_2 < \ldots < t_n,\ x \in X,\ \Gamma_1 \in \mathcal{B}, \ldots, \Gamma_n \in \mathcal{B}$, then

$$\mathbb{P}_{s,x}(\bigcap_{k=1}^{n}\{x(t_k) \in \Gamma_k\}) =$$

$$= \int_{\Gamma_1} P(t_1 - s, x, dy_1) \int_{\Gamma_2} P(t_2 - t_1, y_1, dy_2) \ldots \int_{\Gamma_n} P(t_n - t_{n-1}, y_{n-1}, dy_n) =$$

$$= \mathbb{P}_{0,x}(\bigcap_{k=1}^{n}\{x(t_k - s) \in \Gamma_k\}).$$

For a homogeneous Markov process $(x(t), \mathcal{M}_t^s, \mathbb{P}_{s,x})$ in $(X, \mathcal{B})$, we put $\mathbb{P}_x(\cdot) = \mathbb{P}_{0,x}(\cdot)$ for $x \in X$ and $\mathcal{M}_t = \mathcal{M}_t^0$ for $t \ge 0$. Then the following conditions are fulfilled:

(i^0) $\mathbb{P}_x(\{x(0) = x\}) = 1$ for $x \in X$;

(ii^0) for fixed $t \ge 0$ and $\Gamma \in \mathcal{B}$, the function $P(t, x, \Gamma) = \mathbb{P}_x(\{x(t) \in \Gamma\}), x \in X$, is a $\mathcal{B}$-measurable one;

(iii^0) for all $t \ge 0,\ s \ge 0,\ x \in X$ and $\Gamma \in \mathcal{B}$, the relation

$$\mathbb{P}_x(\{x(s+t) \in \Gamma\}/\mathcal{M}_t) = P(s, x(t), \Gamma)$$

holds true $\mathbb{P}_x$-almost surely.

The notation $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ will be used for a homogeneous Markov process.

1.4. Examples and exercises. In any example below, a certain function $g(t, x, y), t > 0, x \in X$ and $y \in X$, is defined for X being either a d-dimensional Euclidean space $\mathbb{R}^d$ or some part of it with the σ-algebra $\mathcal{B}$ of all Borel measurable subsets of X. The following problem is proposed to the reader: make sure that the function $P(t, x, \Gamma)$ defined for $t > 0, x \in X$ and $\Gamma \in \mathcal{B}$ by the Lebesgue integral

$$P(t, x, \Gamma) = \int_{\Gamma} g(t, x, y)dy$$

can serve as transition probability of a homogeneous continuous Markov process in X. The function g is then called transition probability density of the process.

1.4.A. For $t > 0, x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, we put

$$g_0(t, x, y) = (2\pi t)^{-d/2} \exp\{-|y - x|^2/2t\}.$$

A homogeneous continuous Markov process in $\mathbb{R}^d$ generated by this transition probability density is called Brownian motion or Wiener process.

1.4.B. Let $X = [0, +\infty)$. We set

$$g(t, x, y) = (2\pi t)^{-1/2}[\exp\{-(y - x)^2/2t\} + \exp\{-(y + x)^2/2t\}]$$

for $t > 0, x \ge 0$ and $y \ge 0$. A homogeneous continuous Markov process in X with this transition probability density is called Brownian motion in $\mathbb{R}^1$ reflected at the origin.

1.4.C. For fixed $q \in \mathbb{R}^1$, we put

$$g(t, x, y) = (2\pi t)^{-1/2}[\exp\{-(y - x)^2/2t\} + q \operatorname{sign} y \exp\{-(|y| + |x|)^2/2t\}]$$

for all $t > 0, x \in \mathbb{R}^1, y \in \mathbb{R}^1$ (we believe that $\operatorname{sign} 0 = 0$). This function is discontinuous (if $q \ne 0$) at the point $y = 0$: we have $g(t, x, 0\pm) = \frac{1 \pm q}{\sqrt{2\pi t}} \exp\{-x^2/2t\}$ and $g(t, x, 0) = \frac{1}{\sqrt{2\pi t}} \exp\{-x^2/2t\}$. Nevertheless it satisfies the Kolmogorov–Chapman equation

$$g(s + t, x, y) = \int_{\mathbb{R}^1} g(s, x, z)g(t, z, y)dz,\ \ s > 0, t > 0, x \in \mathbb{R}^1, y \in \mathbb{R}^1,$$

as well as the relation

$$\int_{\mathbb{R}^1} g(t, x, y)dy = 1,\ \ t > 0, x \in \mathbb{R}^1.$$

It remains to observe that all the values of the function g are non-negative iff $q \in [-1, 1]$ and that the continuity condition is fulfilled in this case. Hence, for each $q \in [-1, 1]$ there exists a continuous homogeneous Markov process in $\mathbb{R}^1$ whose transition probability density is given by the function g. If $q=0$, we have one-dimensional Brownian motion. Any process for $q \in [-1,1]\setminus\{0\}$ is called skew Brownian motion.

1.4.D. For a fixed unit vector $\nu \in \mathbb{R}^d$ ($d \geq 2$ in this example), we set $S=\{x \in \mathbb{R}^d : (x,\nu)=0\}$. Let a continuous bounded function $(q(x))_{x\in S}$ taking on its values from the interval $[-1,1]$ be given. Making use of the function g_0 from 1.4.A, we define a function g of the arguments $t>0,\ x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ by setting

$$g(t,x,y)=g_0(t,x,y)+\int_0^t d\tau \int_S g_0(\tau,x,z)\frac{\partial g_0(t-\tau,z,y)}{\partial \nu_z} q(z)d\sigma_z,$$

where the inner integral on the right-hand side is a surface integral (since S is a $(d-1)$-dimensional subspace of $\mathbb{R}^d$, that integral is nothing else but the Lebesgue integral in $\mathbb{R}^{d-1}$) and $\frac{\partial}{\partial \nu_z}$ means the derivative of the function $(g_0(t-\tau,z,y))_{z\in\mathbb{R}^d}$ (for fixed $\tau<t$ and $y \in \mathbb{R}^d$) in the direction ν, that is

$$\frac{\partial g_0(t-\tau,z,y)}{\partial \nu_z}=\frac{(y-z,\nu)}{t-\tau}g_0(t-\tau,z,y),\ 0\leq \tau<t,\ z\in\mathbb{R}^d,\ y\in\mathbb{R}^d.$$

It is clear that $g(t,x,y)=g_0(t,x,y)$ for $y\in S$. In the case of $y\notin S$, the integrals in the formula for g are well-defined as follows from the relations (we use the designations $\|q\|$ for $\sup_{x\in S}|q(x)|$ and $\widetilde{x}$ for the orthogonal projection of $x\in\mathbb{R}^d$ on S, that is $\widetilde{x}=x-\nu(x,\nu)$):

$$\int_0^t d\tau \int_S g_0(\tau,x,z)\Big|\frac{\partial g_0(t-\tau,z,y)}{\partial \nu_z}\Big||q(z)|d\sigma_z \leq$$

$$\leq \|q\|\int_0^t \frac{|(y,\nu)|\exp\{-(x,\nu)^2/2\tau-(y,\nu)^2/2(t-\tau)\}}{\sqrt{2\pi\tau}\sqrt{2\pi(t-\tau)^3}}d\tau\cdot$$

$$\cdot\int_S \frac{\exp\{-|z-\widetilde{x}|^2/2\tau-|\widetilde{y}-z|^2/2(t-\tau)\}}{(2\pi\tau)^{(d-1)/2}(2\pi(t-\tau))^{(d-1)/2}}d\sigma_z=$$

$$=\|q\|\frac{\exp\{-(|(x,\nu)|+|(y,\nu)|)^2/2t\}}{\sqrt{2\pi t}}\ \frac{\exp\{-|\widetilde{y}-\widetilde{x}|^2/2t\}}{(2\pi t)^{(d-1)/2}}\leq$$

$$\leq \|q\|\frac{\exp\{-|y-x|^2/2t\}}{(2\pi t)^{d/2}}=\|q\|g_0(t,x,y).$$

Similar reasons show that

$$\int_0^t d\tau\int_S g_0(\tau,x,z)\frac{\partial g_0(t-\tau,z,y)}{\partial\nu_z}q(z)d\sigma_z\geq -\|q\|g_0(t,x,y).$$

Therefore, for all $t>0,\ x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$, the inequalities

$$(1-\|q\|)g_0(t,x,y)\leq g(t,x,y)\leq(1+\|q\|)g_0(t,x,y)$$

hold true. So, the values of the function g are non-negative because of $\|q\|\leq 1$.

Now, the reader should prove that the function g satisfies the Kolmogorov–Chapman equation and verify that the continuity condition is fulfilled. So, there exists a continuous homogeneous Markov process in $\mathbb{R}^d$ whose transition probability density is given by the function g. This process is called Brownian motion with a membrane located on the hyperplane S; the function $(q(x))_{x\in S}$ is called the permeability coefficient.

1.4.E. Let $X=[0,+\infty)$. For $t>0,\ \rho\in X$ and $r\in X$, we set

$$g(t,\rho,r)=\frac{r}{t}\exp\{-(\rho^2+r^2)/2t\}I_0(\rho r/t),$$

where $(I_0(z))_{z\in\mathbb{R}^1}$ is a modified Bessel function of zero order, that is

$$I_0(z)=\sum_{n=0}^{\infty}(\frac{z}{2})^{2n}/(n!)^2.$$

A continuous homogeneous Markov process in X with the function g as its transition probability density is a representative of the whole class of processes that are called Besselean.

1.5. Comments and references. We keep to Dynkin's point of view on the notion of a Markov process (see [1]), according to which neither an initial instant of time, nor initial location of the process is fixed. On the contrary, every instant of time and every point of the phase space can serve as initial data for the process in usual sense. So, a whole class of Markov processes in usual sense is determined by the Definition 1.1. Everything concerning the distribution of such a process is completely determined by the corresponding transition probability.

The continuity condition was established by E.B.Dynkin [2] and independently by J.R.Kinney [3].

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Lesson 2. Diffusion processes.

2.1. Introduction. In this lesson, the phase space of Markov processes will be a d-dimensional Euclidean space $\mathbb{R}^d$ with the σ-algebra $\mathcal{B}$ of all Borel measurable subsets of $\mathbb{R}^d$. Every Markov process in this space is generated by a certain solution to the Kolmogorov–Chapman equation, that is, such a function $P(s,x,t,\Gamma)$ of the arguments $s \geq 0,\ x \in \mathbb{R}^d,\ t > s$ and $\Gamma \in \mathcal{B}$ that possesses the following properties:

1) it is a $\mathcal{B}$-measurable function of $x \in \mathbb{R}^d$ for fixed $s < t$ and $\Gamma \in \mathcal{B}$;

2) it is a probability measure of $\Gamma \in \mathcal{B}$ for fixed $s < t$ and $x \in \mathbb{R}^d$;

3) the equality $P(s,x,t,\Gamma) = \int_{\mathbb{R}^d} P(\tau,y,t,\Gamma)P(s,x,\tau,dy)$ holds true for all $0 \leq s < \tau < t,\ x \in \mathbb{R}^d$ and $\Gamma \in \mathcal{B}$.

The equality in 3) is called the Kolmogorov–Chapman equation. It expresses a general principle, according to which stochastic systems with the Markov property are evolving in time. The equation is non-linear, and it is a problem how to describe these and those classes of its solutions.

At the end of 1920s, A.N.Kolmogorov noticed that some assumptions on the behavior of the process desired on small intervals of time makes it possible to reduce the Kolmogorov–Chapman equation to a certain linear problem. He managed to point out several classes of Markov processes. One of them became later to be called diffusion processes.

2.2. Definition. Recall that an open ball in $\mathbb{R}^d$ of radius $r > 0$ with its center located at the point $x \in \mathbb{R}^d$ is denoted by $B_r(x)$; its complement to $\mathbb{R}^d$ is designated by $B_r(x)^c$.

Let $P(s,x,t,\Gamma),\ 0 \leq s < t,\ x \in \mathbb{R}^d,\ \Gamma \in \mathcal{B}$, be a solution to the Kolmogorov–Chapman equation. We say that this solution is transition probability of a diffusion process in $(\mathbb{R}^d, \mathcal{B})$ if the following conditions are fulfilled:

A) for all $s \geq 0,\ x \in \mathbb{R}^d$ and $\varepsilon > 0$ the relation

$$\lim_{t \downarrow s} \frac{1}{t-s} \int_{B_\varepsilon(x)^c} P(s,x,t,dy) = 0$$

holds;

B) for all $s \geq 0,\ x \in \mathbb{R}^d$ and some $\varepsilon > 0$ the limit

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{B_\varepsilon(x)} (y-x)P(s,x,t,dy)$$

exists;

C) for all $s \geq 0,\ x \in \mathbb{R}^d,\ \theta \in \mathbb{R}^d$ and some $\varepsilon > 0$ the limit

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{B_\varepsilon(x)} (y-x,\theta)^2 P(s,x,t,dy)$$

exists.

One can easily observe that under the condition A), the existence of the limits in the conditions B) and C) for some $\varepsilon > 0$ implies their existence for any $\varepsilon > 0$, and the fact that those limits do not depend on ε. So, the limit in B) determines an $\mathbb{R}^d$-valued function $a(s,x),\ s \geq 0,\ x \in \mathbb{R}^d$, while the limit in C) determines a linear operator $b(s,x),\ s \geq 0, x \in \mathbb{R}^d$, for which that limit can be written as the quadratic form $(b(s,x)\theta,\theta),\ \theta \in \mathbb{R}^d$; it is clear that this form is non-negative definite. The function $a(s,x),\ s \geq 0,\ x \in \mathbb{R}^d$, is called drift vector and the function $b(s,x), s \geq 0, x \in \mathbb{R}^d$ is called diffusion operator and they all together are called local characteristics of the corresponding diffusion process.

This terminology is connected with the fact that diffusion processes are intended to serve as a mathematical model for describing the motion of a diffusing particle suspended in a liquid or a gas. Such a particle takes part in the motions of two kinds. One of them is caused by some streams in the liquid or winds in the gas. The local velocity of this macroscopic motion is given by the drift vector. The other kind of motion is microscopic. It is the result of collisions between our particle and molecules of the liquid or gas. The diffusion operator characterizes locally the intensity of that molecular motion in different directions.

The integrals in the conditions B) and C) are taken over the balls because any moments of the process a priory do not suppose to exist. But if those moments do exist, then the corresponding integrals can be taken over the whole $\mathbb{R}^d$. In that case, the following conditions are more convenient to be checked in order to verify that a given transition probability determines some diffusion process.

Transition probability $P(s,x,t,dy),\ 0 \leq s < t,\ x \in \mathbb{R}^d$ generates a diffusion process if the following conditions are fulfilled:

A$'$) for some $\delta > 0$ and all $s \geq 0,\ x \in \mathbb{R}^d$, the relation

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{\mathbb{R}^d} |y-x|^{2+\delta} P(s,x,t,dy) = 0$$

holds true;

B$'$) for all $s \geq 0$ and $x \in \mathbb{R}^d$, the limit

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{\mathbb{R}^d} (y-x)P(s,x,t,dy)$$

exists;

C$'$) for all $s \geq 0,\ x \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^d$, the limit

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{\mathbb{R}^d} (y-x,\theta)^2 P(s,x,t,dy)$$

exists.

It is evident that the limits in B') and C') coincide with $a(s,x)$ and $(b(s,x)\theta,\theta)$, respectively.

2.3. Kolmogorov's backward and forward equations. Let transition probability $P(s,x,t,dy),\ 0 \leq s < t,\ x \in \mathbb{R}^d$, satisfying the conditions A)–C) be given. We first prove the following auxiliary result.

Proposition. If $(f(x))_{x\in\mathbb{R}^d}$ is an arbitrary bounded twice continuously differentiable function with real values, then for all $s \ge 0$ and $x \in \mathbb{R}^d$, the equality

$$\lim_{t\downarrow s} \frac{1}{t-s}\Big[\int_{\mathbb{R}^d} f(y)P(s,x,t,dy) - f(x)\Big] = (a(s,x), f'(x)) + \frac{1}{2}\operatorname{Tr}(b(s,x)f''(x)) \tag{1}$$

holds true.

Proof. The boundedness of the function $(f(x))_{x\in\mathbb{R}^d}$ and the condition A) imply the fact that the left-hand side of (1) is equal to the expression

$$\lim_{t\downarrow s} \frac{1}{t-s}\int_{B_\varepsilon(x)} [f(y) - f(x)]P(s,x,t,dy)$$

for all $\varepsilon > 0$. Applying now Taylor's formula and making use of the conditions B) and C), we arrive at the conclusion that this expression can be written as follows

$$(a(s,x), f'(x)) + \frac{1}{2}\operatorname{Tr}(b(s,x)f''(x)) + R_\varepsilon,$$

where absolute value of R_ε for $\varepsilon > 0$ can be estimated by the expression

$$\frac{1}{2}\sup_{z\in B_\varepsilon(x)} \|f''(z) - f''(x)\| \lim_{t\downarrow s}\frac{1}{t-s}\int_{B_\varepsilon(x)} |y-x|^2 P(s,x,t,dy) =$$

$$= \frac{1}{2}\sup_{z\in B_\varepsilon(x)} \|f''(z) - f''(x)\| \cdot \operatorname{Tr}(b(s,x))$$

(we have used here the operator norm of the gessian $f''(\cdot)$). Since the function $(f''(x))_{x\in\mathbb{R}^d}$ is continuous, we have

$$\overline{\lim_{\varepsilon\to 0}}|R_\varepsilon| \le \frac{1}{2}\operatorname{Tr}(b(s,x)) \lim_{\varepsilon\to 0}\sup_{z\in B_\varepsilon(x)} \|f''(z) - f''(x)\| = 0.$$

This completes the proof.

Theorem 1. *Suppose that a given transition probability $P(s,x,t,dy)$, $x \in \mathbb{R}^d$, $0 \le s < t$, corresponds to a diffusion process with its drift vector and diffusion operator being continuous functions. Let $(\varphi(x))_{x\in\mathbb{R}^d}$ be such a bounded continuous function with real values that the function $u(s,x)$ of the arguments $(s,x) \in [0,t)\times\mathbb{R}^d$ (for fixed $t > 0$) defined by*

$$u(s,x) = \int_{\mathbb{R}^d} \varphi(y)P(s,x,t,dy)$$

is twice differentiable with respect to the argument $x \in \mathbb{R}^d$ continuously with respect to the pair (s,x). Then this function is also differentiable with respect to $s \in [0,t)$ and satisfies the equation

$$u'_s(s,x) + (a(s,x), u'_x(s,x)) + \frac{1}{2}\operatorname{Tr}(b(s,x)u''_{xx}(s,x)) = 0 \tag{2}$$

in the domain $(s,x) \in [0,t)\times\mathbb{R}^d$ and also the final condition

$$\lim_{s\uparrow t} u(s,x) = \varphi(x) \tag{3}$$

for all $x \in \mathbb{R}^d$ is fulfilled.

Proof. Let $s \in [0,t)$ and $\Delta s > 0$ be such that $s+\Delta s < t$. Then using the Kolmogorov–Chapman equation, we can write down the equality

$$\frac{1}{\Delta s}[u(s,x) - u(s+\Delta s,x)] = \frac{1}{\Delta s}\int_{\mathbb{R}^d}[u(s+\Delta s,z) - u(s+\Delta s,x)]P(s,x,s+\Delta s,dz).$$

Applying the proposition to the function $(u(s+\Delta s,x))_{x\in\mathbb{R}^d}$ leads us to the equation (2).

Notice now that for $x \in \mathbb{R}^d$ and any $\varepsilon > 0$, we have

$$\lim_{s\uparrow t} |u(s,x) - \varphi(x)| \le \lim_{s\uparrow t} \int_{B_\varepsilon(x)} |\varphi(y) - \varphi(x)| P(s,x,t,dy) \le \sup_{y\in B_\varepsilon(x)} |\varphi(y) - \varphi(x)|.$$

The last quantity converges to 0, as $\varepsilon \downarrow 0$ for fixed $x \in \mathbb{R}^d$. The theorem has been proved.

Equation (2) is called Kolmogorov's backward equation. It is a linear second order partial differential equation of parabolic type (for fixed $s \ge 0$ and $x \in \mathbb{R}^d$, the operator $b(s,x)$ is non-negative definite). We have thus seen that the assumption on the process to be a diffusion one plus some additional conditions allow us to linearize the Kolmogorov–Chapman equation.

Let an orthonormal basis in $\mathbb{R}^d$ be fixed. Denote by x^j for $j \in \{1,2,\dots,d\}$ the coordinates of $x \in \mathbb{R}^d$ in that basis. The matrix of the operator $b(s,x)$ in that basis consists of the entries denoted by $b_{jk}(s,x)$ for $j,k \in \{1,2,\dots,d\}$. The Kolmogorov backward equation in coordinates form is written as follows

$$\frac{\partial u}{\partial s}(s,x) + \sum_{j=1}^d a^j(s,x) \frac{\partial u}{\partial x^j}(s,x) + \frac{1}{2} \sum_{j,k=1}^d b_{jk}(s,x) \frac{\partial^2 u}{\partial x^j \partial x^k}(s,x) = 0. \qquad (2')$$

Suppose now that transition probability of a diffusion process is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^d$, that is

$$P(s,x,t,\Gamma) = \int_\Gamma G(s,x,t,y)dy, \ 0 \le s < t, \ x \in \mathbb{R}^d, \ \Gamma \in \mathcal{B}.$$

The function G is called transition probability density of the process. It turns out that under some conditions, the function $G(s,x,t,y)$ as a function of the arguments $t \in (s,+\infty)$ and $y \in \mathbb{R}^d$ (for fixed $s \ge 0$ and $x \in \mathbb{R}^d$) satisfies some partial differential equation that is formally conjugate to equation $(2')$.

Theorem 2. *Suppose that transition probability density $G(s,x,t,y)$, $0 \le s < t, x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, of a diffusion process is such that the limits in conditions B) and C) exist locally uniformly with respect to $x \in \mathbb{R}^d$ and let the following derivatives exist and be continuous in $(t,y) \in (s,+\infty) \times \mathbb{R}^d$*

$$\frac{\partial G(s,x,t,y)}{\partial t}, \ \frac{\partial (a^j(t,y)G(s,x,t,y))}{\partial y^j}, \ \frac{\partial^2 (b_{jk}(t,y)G(s,x,t,y))}{\partial y^j \partial y^k}$$

for all j and k from the set $\{1,2,\dots,d\}$. Then for fixed $s \ge 0$ and $x \in \mathbb{R}^d$, the function $G(s,x,t,y)$, $(t,y) \in (s,+\infty) \times \mathbb{R}^d$, satisfies the following equation

$$\frac{\partial G(s,x,t,y)}{\partial t} + \sum_{j=1}^d \frac{\partial (a^j(t,y)G(s,x,t,y))}{\partial y^j} - \frac{1}{2} \sum_{j,k=1}^d \frac{\partial^2 (b_{jk}(t,y)G(s,x,t,y))}{\partial y^j \partial y^k} = 0. \qquad (4)$$

Proof. Let $(\varphi(x))_{x\in\mathbb{R}^d}$ be a real-valued compactly supported and twice continuously differentiable function. We have

$$\int_{\mathbb{R}^d} \varphi(y) \frac{\partial G(s,x,t,y)}{\partial t} dy =$$

$$= \lim_{\Delta t\downarrow 0} \frac{1}{\Delta t} \Big[\int_{\mathbb{R}^d} \varphi(y) G(s,x,t+\Delta t,y)dy - \int_{\mathbb{R}^d} \varphi(y) G(s,x,t,y)dy \Big] =$$

$$= \lim_{\Delta t\downarrow 0} \frac{1}{\Delta t} \int_{\mathbb{R}^d} G(s,x,t,y) \Big[\int_{\mathbb{R}^d} G(t,y,t+\Delta t,z)(\varphi(z) - \varphi(y))dz \Big] dy.$$

Our proposition and the assumptions of Theorem 2 allow us to pass to the limit here. As a result, we obtain the equality

$$\int_{\mathbb{R}^d} \varphi(y) \frac{\partial G(s,x,t,y)}{\partial t} dy = \int_{\mathbb{R}^d} G(s,x,t,y)[(a(t,y),\varphi'(y)) + \frac{1}{2}\operatorname{Tr}(b(t,y)\varphi''(y))]dy.$$

Integrating here by part leads us to equation (4). The theorem has just been proved.

Equation (4) is called Kolmogorov's forward equation. In the literature closer to physics, it is called the Fokker–Planck equation.

The results of Kolmogorov described above indicate a path by which one may hope to solve the problem of the existence of a diffusion process with previously specified its local characteristics. The main station on this path is the investigation of the Cauchy problem (2) – (3). In the next lesson some classical results on the existence and uniqueness of a solution to this Cauchy problem will be formulated.

Remark. All the results formulated above can be obviously reformulated in the case of a diffusion process being homogeneous.

In particular, the local characteristics of a homogeneous diffusion process (that is, its drift vector and its diffusion operator) do not depend on time. If $P(t, x, dy)$, $t > 0, x \in \mathbb{R}^d$, is transition probability of such a process, then under some assumptions on a given continuous bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$, the function $u(t,x) = \int_{\mathbb{R}^d} \varphi(y)P(t,x,dy)$, $t > 0,\ x \in \mathbb{R}^d$, satisfies the equation

$$u'_t(t,x) - (a(x), u'_x(t,x)) - \frac{1}{2}\,\mathrm{Tr}(b(x)u''_{xx}(t,x)) = 0 \qquad (2^0)$$

in the domain $(t,x) \in (0,+\infty) \times \mathbb{R}^d$ and also the initial condition

$$\lim_{t\downarrow 0} u(t,x) = \varphi(x) \qquad (3^0)$$

for all $x \in \mathbb{R}^d$ is fulfilled.

2.4. Examples and exercises.

2.4.A. Make sure that Brownian motion in $\mathbb{R}^d$ (see 1.4.A) is a diffusion process with $a(x) \equiv 0$ and $b(x) \equiv I$, where I is an identity operator in $\mathbb{R}^d$. Verify that for any real-valued continuous bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$, the function

$$u(t,x) = \int_{\mathbb{R}^d} \varphi(y)g_0(t,x,y)dy,\ t > 0,\ x \in \mathbb{R}^d,$$

is a solution of the Cauchy problem (2^0)–(3^0) (equation (2^0) with $a(x) \equiv 0$ and $b(x) \equiv I$ is called the heat equation).

2.4.B. Let $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ be a homogeneous continuous Markov process in $\mathbb{R}^1$ called skew Brownian motion (see 1.4.C). Verify that the following equalities

$$\int_{\mathbb{R}^1} (y-x)g(t,x,y)dy = q\int_0^t \exp\{-x^2/2\tau\}\frac{d\tau}{\sqrt{2\pi\tau}},$$

$$\int_{\mathbb{R}^1} (y-x)^2 g(t,x,y)dy = t - 2qx\int_0^t \exp\{-x^2/2\tau\}\frac{d\tau}{\sqrt{2\pi\tau}}$$

are fulfilled for all $t > 0$ and $x \in \mathbb{R}^1$. Show that for any $x \in \mathbb{R}^1$

$$\lim_{t\downarrow 0}\frac{1}{t}\int_{\mathbb{R}^1} (y-x)^2 g(t,x,y)dy = 1,$$

$$\lim_{t\downarrow 0}\frac{1}{t}\int_{\mathbb{R}^1} (y-x)g(t,x,y)dy = q\delta(x),$$

where $(\delta(x))_{x\in\mathbb{R}^1}$ is Dirac's δ-function. This means that skew Brownian motion for $q \neq 0$ is not a diffusion process in the sense of definition in Section 2.2, but it can be treated as a diffusion process in some generalized sense. The same concerns also the next example.

2.4.C. Consider Brownian motion in $\mathbb{R}^d$ with a membrane on a given hyperplane (see 1.4.D; we use here the notation from there). Prove that for all $t > 0, x \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^d$, the following relations

$$\int_{\mathbb{R}^d} (y-x,\theta)g(t,x,y)dy = (\nu,\theta)\int_0^t d\tau\int_S g_0(\tau,x,y)q(y)d\sigma_y,$$

$$\int_{\mathbb{R}^d}(y-x,\theta)^2 g(t,x,y)dy = t|\theta|^2 + 2(\nu,\theta)\int_0^t d\tau \int_S g_0(\tau,x,y)(y-x,\theta)q(y)d\sigma_y$$

are held. Show that these relations imply the following ones

$$\lim_{t\downarrow 0}\frac{1}{t}\int_{\mathbb{R}^d}(y-x,\theta)^2 g(t,x,y)dy = |\theta|^2,$$

$$\lim_{t\downarrow 0}\frac{1}{t}\int_{\mathbb{R}^d}(y-x,\theta)g(t,x,y)dy = (\nu,\theta)q(x)\delta_S(x),$$

where $(\delta_S(x))_{x\in\mathbb{R}^d}$ is a generalized function whose action on a test function $(\varphi(x))_{x\in\mathbb{R}^d}$ is given as follows $\langle\delta_S,\varphi\rangle = \int_S \varphi(x)d\sigma$.

2.5. Comments and references. A.N.Kolmogorov in [4] pointed out a class of Markov processes that became later known as diffusion processes. In that paper, the backward and forward equations were derived. The first theorems on the existence and uniqueness were obtained by W.Feller [5]. Since that time many articles and books have been devoted to the theory of diffusion processes, for example [6], [7], [8], [9], [10].

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Lesson 3. The Kolmogorov backward equation.

3.1. Introduction. The main question our lessons address is: what conditions must be imposed on given functions $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ and $(b(s,x))_{(s,x)\in\mathcal{D}_T}$ with their values in $\mathbb{R}^d$ and $\mathcal{L}^+(\mathbb{R}^d)$, respectively, so that these functions can serve as the local characteristics of a diffusion process in $\mathbb{R}^d$ (we have just used the notation $\mathcal{D}_T$ for the set $[0,T]\times\mathbb{R}^d$ and $\mathcal{L}^+(\mathbb{R}^d)$ for the set of all linear symmetric operators on $\mathbb{R}^d$ being non-negative definite). Kolmogorov's results expounded in the previous lesson inspire us with the ideas of looking for the process desired among solutions of the Cauchy problem (2)–(3) of Lesson 2. Fortunately, in the theory of partial differential equations of parabolic type there are some assertions that formulate exact conditions on those given functions that guarantee the existence of the so-called fundamental solution of the associated Kolmogorov backward equation. With the help of that solution a classical solution to the Cauchy problem for that equation can be constructed and it turns out to be unique in a certain class of functions. All these results allow one to conclude that there exists a diffusion process in $\mathbb{R}^d$ whose transition probability density coincides with the fundamental solution mentioned above. We call such a process classical diffusion. It is clear that any notion of a generalized (non-classical) solution to the Cauchy problem

(2)–(3) of Lesson 2 may determine a process that is diffusion only in some generalized sense. Two examples of such a kind can be found in the previous lessons.

3.2. Fundamental solutions. We are given by an $\mathbb{R}^d$-valued function $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ and an $\mathcal{L}^+(\mathbb{R}^d)$-valued function $(b(s,x))_{(s,x)\in\mathcal{D}_T}$. Fix an orthonormal basis in $\mathbb{R}^d$ and denote by $a^j(s,x)$ for $j=1,2,\dots,d$ the coordinates of the vector $a(s,x)$ in that basis and by $b_{jk}(s,x)$ for j and k from the set $\{1,2,\dots,d\}$ the entries of the matrix of the operator $b(s,x)$ in the same basis. Suppose that these functions possess the following properties:

(i) there exist constants c_1 and c_2, $0<c_1\le c_2$, such that for all $\theta\in\mathbb{R}^d$ and $(s,x)\in\mathcal{D}_T$, the inequalities

$$c_1|\theta|^2 \le (b(s,x)\theta,\theta) \le c_2|\theta|^2$$

are held;

(ii) for all $(s,x)\in\mathcal{D}_T$, $(t,y)\in\mathcal{D}_T$ and integers j and k from the set $\{1,2,\dots,d\}$, the inequality

$$|b_{jk}(s,x)-b_{jk}(t,y)| \le K(|y-x|^\alpha + |t-s|^{\alpha/2})$$

holds true with some constants $\alpha\in(0,1]$ and $K>0$;

(iii) for all $j\in\{1.2.\dots,d\}$ the function $(a^j(s,x))_{(s.x)\in\mathcal{D}_T}$ is continuous bounded and satisfies the inequality

$$|a^j(s,x)-a^j(s,y)| \le K|y-x|^\alpha,\ s\in[0,T],\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d.$$

Under these conditions, the so-called fundamental solution $g(s,x,t,y)$, $0\le s<t\le T$, $x\in\mathbb{R}^d, y\in\mathbb{R}^d$, of the equation

$$\frac{\partial u}{\partial s}(s,x)+\sum_{j=1}^d a^j(s,x)\frac{\partial u}{\partial x^j}(s,x)+\frac{1}{2}\sum_{j,k=1}^d b_{jk}(s,x)\frac{\partial^2 u}{\partial x^j\partial x^k}(s,x)=0 \tag{1}$$

exists, as is formulated in Theorem 1 below. We now define the notion of a fundamental solution of equation (1).

Definition. A continuous function $g(s,x,t,y)$ of the arguments $(s,x)\in\mathcal{D}_T$ and $(t,y)\in\mathcal{D}_T$ for $s<t$ is called a fundamental solution of equation (1) if as a function of (s,x) for fixed (t,y) it satisfies equation (1) in the domain $(s,x)\in[0,t)\times\mathbb{R}^d$ and for an arbitrary continuous bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$, the relation

$$\lim_{s\uparrow t}\int_{\mathbb{R}^d} g(s,x,t,y)\varphi(y)dy=\varphi(x),\ x\in\mathbb{R}^d, \tag{2}$$

holds true.

Theorem 1. *Let an $\mathbb{R}^d$-valued function $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ and an $\mathcal{L}^+(\mathbb{R}^d)$-valued function $(b(s,x))_{(s,x)\in\mathcal{D}_T}$ be given. Suppose that they satisfy the conditions (i)–(iii). Then there exists a fundamental solution $g(s,x,t,y)$, $0\le s<t\le T$, $x\in\mathbb{R}^d, y\in\mathbb{R}^d$, of equation (1) satisfying the inequality*

$$|D_s^l D_x^m g(s,x,t,y)| \le L(t-s)^{-\frac{d+2l+m}{2}}\exp\{-\mu\frac{|y-x|^2}{t-s}\} \tag{3}$$

for all $0\le s<t\le T$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$ with some positive constants $L>0$, $\mu>0$ and non-negative integers l and m being such that $2l+m\le 2$; here D_s^l means the derivative of the order l with respect to the argument s and D_x^m means any partial derivative of the order m with respect to the argument x.

Remark. The so-called parametrix method for constructing the fundamental solution in Theorem 1 is used. And it turns out that the constant μ in (3) can be arbitrarily chosen from the interval $(0,\varkappa/2T)$, where $\varkappa$ is defined as follows

$$\varkappa=\inf_{(s,x)\in\mathcal{D}_T}\min_{\theta\in\mathbb{R}^d,|\theta|=1}(b(s,x)^{-1}\theta,\theta).$$

The conditions of Theorem 1 imply the inequality $\varkappa > 0$. Therefore, if a continuous real-valued function $(\varphi(x))_{x\in\mathbb{R}^d}$ satisfies the inequality

$$|\varphi(x)| \le B\exp\{\beta|x|^2\},\ x\in\mathbb{R}^d,$$

with some constants $B>0$ and $\beta\in(0,\varkappa/2T)$, then the integral

$$\int_{\mathbb{R}^d}\varphi(y)g(s,x,t,y)dy$$

exists for all $0\le s<t\le T$ and $x\in\mathbb{R}^d$. As a function of the arguments $(s,x)\in[0,t)\times\mathbb{R}^d$, this integral determines a solution to the Cauchy problem (1)–(2). Moreover, the following estimation

$$|\int_{\mathbb{R}^d} g(s,x,t,y)\varphi(y)dy| \le \mathrm{const}\cdot\exp\{\gamma|x|^2\}$$

holds true for all $0\le s<t\le T$ and $x\in\mathbb{R}^d$ with some constant $\gamma>0$ depending only on $\varkappa,\beta$ and T.

3.3. The maximum principle. One of the forms of the maximum principle for the second order partial differential equations of parabolic type is formulated in the next assertion (the designation $\mathcal{D}_T^0$ is used for the set $[0,T)\times\mathbb{R}^d$).

Theorem 2. *Assume that the coefficients of equation (1) are continuous bounded functions in $\mathcal{D}_T^0$ and let the condition*

$$\sum_{j,k=1}^d b_{jk}(s,x)\theta^j\theta^k \ge 0 \tag{4}$$

be held for all $(s,x)\in\mathcal{D}_T^0$ and all real numbers $\theta^1,\theta^2,\dots,\theta^d$. Suppose further that a continuous real-valued function $(u(s,x))_{(s,x)\in\mathcal{D}_T}$ satisfies the following inequalities:

a)

$$\frac{\partial u}{\partial s}(s,x)+\sum_{j=1}^d a^j(s,x)\frac{\partial u}{\partial x^j}(s,x)+\frac{1}{2}\sum_{j,k=1}^d b_{jk}(s,x)\frac{\partial^2 u}{\partial x^j\partial x^k}(s,x)\le 0,\ (s,x)\in\mathcal{D}_T^0;$$

b) $u(s,x)\ge -B\exp\{\beta|x|^2\},\ (s,x)\in\mathcal{D}_T$, with some positive constants B and β;

c) $u(T,x)\ge 0,\ x\in\mathbb{R}^d$.

Then $u(s,x)\ge 0$ for all $(s,x)\in\mathcal{D}_T$.

Corollary. Under the conditions of Theorem 1, the fundamental solution g of equation (1) takes on only non-negative values.

3.4. The uniqueness theorem. Let continuous functions $(f(s,x))_{(s,x)\in\mathcal{D}_T}$ and $(\varphi(x))_{x\in\mathbb{R}^d}$ with real values be given. Fix some $t\in(0,T]$ and consider the equality

$$\frac{\partial u}{\partial s}(s,x)+\sum_{j=1}^d a^j(s,x)\frac{\partial u}{\partial x^j}(s,x)+\frac{1}{2}\sum_{j,k=1}^d b_{jk}(s,x)\frac{\partial^2 u}{\partial x^j\partial x^k}(s,x)=-f(s,x) \tag{5}$$

in the domain $(s,x)\in[0,t)\times\mathbb{R}^d$ and the final condition

$$\lim_{s\uparrow t}u(s,x)=\varphi(x) \tag{6}$$

for all $x\in\mathbb{R}^d$. One of the consequences from the maximum principle is the following statement on the uniqueness of a solution to the Cauchy problem (5) – (6).

Theorem 3. *Assume that the coefficients of equation (5) are continuous bounded functions in the region $\mathcal{D}_T$ and let inequality (4) be fulfilled for all $(s,x)\in\mathcal{D}_T$ and all real numbers $\theta^1,\theta^2,\dots,\theta^d$. Then in the class of functions satisfying the inequality*

$$|u(s,x)|\le B\exp\{\beta|x|^2\},\ (s,x)\in[0,t]\times\mathbb{R}^d,$$

with some positive constants B and β, there exists no more than one solution to the Cauchy problem (5) – (6).

Corollary. Under the conditions of Theorem 1, the fundamental solution g satisfies the relations:

$$\int_{\mathbb{R}^d} g(s,x,t,y)dy = 1 \tag{7}$$

for all $0 \le s < t \le T$ and $x \in \mathbb{R}^d$;

$$\int_{\mathbb{R}^d} g(s,x,\tau,z)g(\tau,z,t,y)dz = g(s,x,t,y) \tag{8}$$

for all $0 \le s < \tau < t \le T,\ x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$.

This corollary together with that of Section 3.3 show that under the conditions of Theorem 1, the fundamental solution $g(s,x,t,y),\ 0 \le s < t \le T,\ x \in \mathbb{R}^d, y \in \mathbb{R}^d$, is the transition probability density of a Markov process in $\mathbb{R}^d$. Inequality (3) for $l = 0$ and $m = 0$ allows us assert that the continuity conddition from Lesson 1 is fulfilled for this process. A question arises, whether this process is diffusion in Kolmogorov's sense or not. We will answer this question in the next section.

3.5. Solving the Cauchy problem (5)–(6). An $\mathbb{R}^d$-valued function $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ and $\mathcal{L}^+(\mathbb{R}^d)$-valued function $(b(s,x))_{(s,x)\in\mathcal{D}_T}$ satisfying the conditions of Theorem 1 are assumed to be given. As follows from Section 3.2, a solution to the homogeneous Cauchy problem (5) – (6) (that is, with $f(s,x) \equiv 0$) in the domain $(s,x) \in [0,t) \times \mathbb{R}^d$ ($t \in (0,T]$ is fixed) can be given by the integral

$$u_0(s,x) = \int_{\mathbb{R}^d} g(s,x,t,y)\varphi(y)dy$$

if a given continuous function $(\varphi(x))_{x\in\mathbb{R}^d}$ satisfies the inequality $|\varphi(x)| \le B\exp\{\beta|x^2|\}$ for all $x \in \mathbb{R}^d$ with some constants $B > 0$ and $\beta \in (0, \varkappa/2T)$. It thus remains to find out a solution to the Cauchy problem (5) – (6) with $\varphi(x) \equiv 0$. Assuming that the function $(f(s,x))_{(s,x)\in\mathcal{D}_T}$ satisfies the inequality $|f(s,x)| \le B\exp\{\beta|x^2|\}$ with the same constants as above, consider the integral

$$u_1(s,x) = \int_s^t d\tau \int_{\mathbb{R}^d} g(s,x,\tau,z)f(\tau,z)dz,\ (s,x) \in [0,t) \times \mathbb{R}^d.$$

Using inequality (3) for $l = 0, m = 0$ and $l = 0, m = 1$, we can assert that the function $u_1(s,x)_{(s,x)\in\mathcal{D}_t^0}$ is continuous and continuously differentiable with respect to $x \in \mathbb{R}^d$. As for the second derivatives with respect to x and the first derivative with respect to s, their existences can be guaranteed under the following assumption on the function f : for any $x \in \mathbb{R}^d$ there exists such $\delta > 0$ that for all $s \in [0,T]$ and $y \in B_\delta(x)$, the inequality

$$|f(s,y) - f(s,x)| \le K|x-y|^\gamma$$

holds true with some constants $K > 0$ and $\gamma \in (0,1]$. In this case, the following formulae are fulfilled

$$\frac{\partial^2 u_1}{\partial x^j \partial x^k}(s.x) = \int_s^t d\tau \int_{\mathbb{R}^d} \frac{\partial^2 g}{\partial x^j \partial x^k}(s,x,\tau,z)f(\tau,z)dz,$$

$$\frac{\partial u_1}{\partial s}(s,x) = -f(s,x) + \int_s^t d\tau \int_{\mathbb{R}^d} \frac{\partial g}{\partial s}(s,x,\tau,z)f(\tau,z)dz,$$

as it follows from the equalities

$$\int_{\mathbb{R}^d} D_s g(s,x,t,y)dy \equiv 0 \text{ and } \int_{\mathbb{R}^d} D_x^m g(s,x,t,y)dy \equiv 0$$

for $m = 1$ and $m = 2$.

As a consequence, we have the following assertion.

Theorem 4. *Assume that the coefficients of equation (5) satisfy the conditions of Theorem 1 and let given real-valued continuous functions* $(f(s,x))_{(s,x)\in\mathcal{D}_T}$ *and* $(\varphi(x))_{x\in\mathbb{R}^d}$ *satisfy the inequalities*

$$|f(s,x)| \le B\exp\{\beta|x|^2\},\ \ |\varphi(x)| \le B\exp\{\beta|x|^2\}$$

for all $s\in[0,T]$ *and* $x\in\mathbb{R}^d$ *with some positive constants* B *and* β. *Suppose, in addition, that* $(f(s,x))_{(s,x)\in\mathcal{D}_T}$ *is locally Hölder continuous in* $x\in\mathbb{R}^d$ *uniformly with respect to* $s\in[0.T]$. *Then a solution to the Cauchy problem (5) – (6) can be written as follows*

$$u(s,x) = \int_{\mathbb{R}^d} g(s,x,t,y)\varphi(y)dy + \int_s^t d\tau \int_{\mathbb{R}^d} g(s,x,\tau,z)f(\tau,x)dz,\ \ (s,x)\in\mathcal{D}_t^0.$$

Moreover, this solution satisfies the inequality

$$|u(s,x)| \le \mathrm{const}\exp\{\gamma|x|^2\},\ \ (s,x)\in\mathcal{D}_t,$$

with some constant $\gamma>0$.

Corollary. Under the assumptions of Theorem 1, the following relations are held for all $0\le s<t\le T, x\in\mathbb{R}^d$ and $\theta\in\mathbb{R}^d$:

$$\int_{\mathbb{R}^d}(y,\theta)g(s,x,t,y)dy = (x,\theta) + \int_s^t d\tau\int_{\mathbb{R}^d}(a(\tau,y),\theta)g(s,x,\tau,y)dy, \tag{9}$$

$$\int_{\mathbb{R}^d}(y,\theta)^2 g(s,x,t,y)dy = (x,\theta)^2 + \int_s^t d\tau\int_{\mathbb{R}^d}(b(\tau,y)\theta,\theta)g(s,x,\tau,y)dy+$$

$$+2\int_s^t d\tau\int_{\mathbb{R}^d}(a(\tau,y),\theta)(y,\theta)g(s,x,\tau,y)dy. \tag{10}$$

Proof consists in verifying the fact that the expression on the left-hand side of (9) (or (10)) solves the same Cauchy problem (5)–(6) as the expression on the right-hand side of (9) (or (10)) does. To verify this is proposed for the reader as an exercise.

One more exercise consists in proving the following relations being simple consequences of (9) and (10):

$$\int_{\mathbb{R}^d}(y-x,\theta)g(s,x,t,y)dy = \int_s^t d\tau\int_{\mathbb{R}^d} g(s,x,\tau,z)(a(\tau,z),\theta)dz \tag{9$'$}$$

$$\int_{\mathbb{R}^d}(y-x,\theta)^2 g(s,x,t,y)dy = \int_s^t d\tau\int_{\mathbb{R}^d} g(s,x,\tau,z)(b(\tau,z)\theta,\theta)dz+$$

$$+2\int_s^t d\tau\int_{\mathbb{R}^d} g(s,x,\tau,z)(z-x,\theta)(a(\tau,z),\theta)dz. \tag{10$'$}$$

The final exercise in this lesson consists in proving the following statement.

Theorem 5. *Let an* $\mathbb{R}^d$*-valued function* $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ *and an* $\mathcal{L}^+(\mathbb{R}^d)$*-valued function* $(b(s,x))_{(s,x)\in\mathcal{D}_T}$ *be given such that they satisfy the conditions (i)–(iii). Then the fundamental solution* $g(s,x,t,y), 0\le s<t\le T, x\in\mathbb{R}^d, y\in\mathbb{R}^d$ *of equation (1) can serve as transition probability density of a diffusion process in* $\mathbb{R}^d$ *whose local characteristics coincide with those given functions.*

3.6. Comments and references. The parametrix method for constructing a fundamental solution of equation (1) is expounded in many books and papers, for example, [1], [13], [14]. The reader can also find there various versions of the maximum principle for that equation. The theorems of this section are taken mainly from [12].

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Lesson 4. Diffusion processes in irregular media.

4.1. Introduction. Let $(x(t))_{t\in[0,T]}$ be a classical diffusion process in $\mathbb{R}^d$ with its local characteristics being given functions $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ and $(b(s,x))_{(s,x)\in\mathcal{D}_T}$ respectively, $\mathbb{R}^d$-valued and $\mathcal{L}^+(\mathbb{R}^d)$-valued. Suppose that the vector field $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ generates a dynamical system $(x_0(t))_{t\in[0,T]}$ in the following sense: for every $s\in[0,T)$ and $\xi\in\mathbb{R}^d$, we have

$$\dot{x}_0(t)=a(t,x_0(t)),\ t\in(s,T],\ x_0(s)=\xi. \tag{1}$$

In this case, the process $(x(t))_{t\in[0,T]}$ can be considered as the result of perturbing the dynamical system $(x_0(t))_{t\in[0,T]}$ by some random factors that are described by the operator-field $(b(s,x))_{(s,x)\in\mathcal{D}_T}$. Notice that those random factors generate a diffusion process $(\xi(t))_{t\in[0,T]}$ in $\mathbb{R}^d$ whose transition probability density $g(s,x,t,y), 0\le s<t\le T, x\in\mathbb{R}^d, y\in\mathbb{R}^d$, is a fundamental solution of the equation

$$u'_s(s,x)+\frac{1}{2}\operatorname{Tr}(b(s,x)u''_{xx}(s,x))=0. \tag{2}$$

An opposite view-point on the process $(x(t))_{t\in[0,T]}$ consists in considering it as the result of perturbing the process $(\xi(t))_{t\in[0,T]}$ by the vector-field $(a(s,x))_{(s,x)\in\mathcal{D}_T}$. It turns out that such a perturbation can be fulfilled for a vector-field that itself does not generate any dynamical system of the kind (1). This lesson will be devoted to locally unbounded perturbing vector-fields. In the next one we will see that such a perturbation is possible even for generalized functions of a certain class.

4.2. Perturbation formulae. We use the notation of Introduction. Suppose that the Markov processes $(x(t))_{t\in[0,T]}$ and $(\xi(t))_{t\in[0,T]}$ in $\mathbb{R}^d$ are classical diffusion. Transition probability density $g(s,x,t,y), 0\le s<t\le T, x\in\mathbb{R}^d, y\in\mathbb{R}^d$, of the process $(\xi(t))_{t\in[0,T]}$ coincides with the fundamental solution of equation (2). Denote by $G(s,x,t,y)$ for $0\le s<t\le T, x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$ transition probability density of the process $(x(t))_{t\in[0,T]}$. It coincides with the fundamental solution of the equation

$$U'_s(s,x)+(a(s,x),U'_x(s,x))+\frac{1}{2}\operatorname{Tr}(b(s,x)U''_{xx}(s,x))=0. \tag{3}$$

Theorem 4 from the previous lesson implies the following relations between the functions g and G.

Theorem 1. *For all $0\le s<t\le T, x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$, the relations*

$$G(s,x,t,y)=g(s,x,t,y)+\int_s^t d\tau\int_{\mathbb{R}^d}g(s,x,\tau,z)(a(\tau,z),G'_z(\tau,z,t,y))dz \tag{4}$$

$$G(s,x,t,y) = g(s,x,t,y) + \int_s^t d\tau \int_{\mathbb{R}^d} G(s,x,\tau,z)(a(\tau,z), g'_z(\tau,z,t,y))dz$$

are held true.

Each one of these equalities can be treated as an equation for the function G if the function g is known. Of course, if the function $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ satisfies the conditions of Theorem 1 from the previous lesson, then the function G can be constructed by the parametrix method for equation (3). But any one of equations (4) can be solved under the assumptions on the function $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ being not so strong.

Equations (4) are known in mathematics as the perturbations formulae. The first one is an analogy to Kolmogorov's backward equation, while the second one is analogous to the Kolmogorov forward equation.

4.3. Diffusion processes with integrable drift vector. We are going to construct a solution to the second one of equations (4) under the following assumptions on a given $\mathcal{L}^+(\mathbb{R}^d)$-valued function $(b(s,x))_{(s,x)\in\mathcal{D}_T}$ and a given $\mathbb{R}^d$-valued function $(a(s,x))_{(s,x)\in\mathcal{D}_T}$: the first one is supposed to satisfy the conditions (i)–(ii) of Lesson 3 and the second one is supposed to be measurable and such that

$$\|a\|_{p,T} = \Big(\int_0^T d\tau \int_{\mathbb{R}^d} |a(\tau,z)|^p dz\Big)^{1/p} < +\infty \tag{5}$$

for some $p > d+2$. We will see that the solution $G(s,x,t,y)$ for $0 \le s < t \le T,\ x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is continuous and continuously dufferentiable with respect to x, but the existence of the second derivatives is not guaranteed. So, the function G can be called a generalized fundamental solution to equation (3) and the corresponding Markov process turns out to be a diffusion one only in some generalized sense.

The following auxiliary result will be useful in constructing a solution to the second equation in (4). Its proof is elementary. For fixed $T > 0, C > 0, \mu > 0$ and $\beta \in \mathbb{R}^1$, let $H_T(C,\mu,\beta)$ be the designation for the class of all real-valued continuous functions $h(s,x,t,y)$ defined for $0 \le s < t \le T,\ x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ and being such that

$$|h(s,x,t,y)| \le C(t-s)^{-\beta} \exp\{-\mu|y-x|^2/(t-s)\}$$

for all $0 \le s < t \le T,\ x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$.

Lemma 1. *Assume that $h_k \in H_T(C_k,\mu,\beta_k)$ for $k \in \{1,2\}$ are given functions, where $\mu > 0, C_k > 0$ and $\beta_k \le \frac{1}{2}(d+1), k \in \{1,2\}$, and let $(f(s,x))_{(s,x)\in\mathcal{D}_T}$ be a real-valued function such that*

$$\|f\|_{p,T} = \Big(\int_0^T \int_{\mathbb{R}^d} |f(s,x)|^p ds dx\Big)^{1/p} < +\infty$$

for some $p > d+2$. Then a function $h(s,x,t,y)$ defined for $0 \le s < t \le T,\ x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ by the integral

$$h(s,x,t,y) = \int_s^t d\tau \int_{\mathbb{R}^d} h_1(s,x,\tau,z) h_2(\tau,z,t,y) f(\tau,z) dz$$

belongs to the class $H_T(C\|f\|_{p,T},\mu,\beta)$, where $\beta = \beta_1 + \beta_2 - (d+2)/2q$,

$$C = C_1 C_2 (\frac{\pi}{\mu q})^{d/2q} [\mathrm{B}(\frac{1}{2}(d+2) - q\beta_1, \frac{1}{2}(d+2) - q\beta_2)]^{1/q},\ q = p/(p-1)$$

($\mathrm{B}(\gamma,\delta)$ means Euler's beta-function).

We now apply the method of successive approximations for solving the second equation in (4). We put $G_0(s,x,t,y) = g(s,x,t,y)$ and for $k \ge 1$

$$G_k(s,x,t,y) = \int_s^t d\tau \int_{\mathbb{R}^d} G_{k-1}(s,x,\tau,z)(a(\tau,z), \nabla_z g(\tau,z,t,y))dz.$$

Remind that the given functions a and b are assumed to satisfy the conditions formulated at the very beginning of this section. Using Lemma 1 and induction on k, we arrive at the following estimation

$$|G_k(s,x,t,y)| \leq L\Big[\frac{\Gamma(\gamma q+q/2)}{\Gamma((k+1)\gamma q+q/2)}\Big]^{1/q} M^k(t-s)^{-d/2+k\gamma}\exp\Big\{-\frac{\mu|y-x|^2}{t-s}\Big\} \quad (6)$$

valid for all $0 \leq s < t \leq T,\ x\in\mathbb{R}^d, y\in\mathbb{R}^d$ and $k=0,1,\dots,$ where $q=p/(p-1), \gamma=(p-d-2)/2p$, $M = L\|a\|_{p,T}[\Gamma(\gamma q)(\frac{\pi}{\mu q})^{d/2}]^{1/q}$ and the positive constants μ and L are taken from the inequalities (see Theorem 1 in Lesson 3)

$$g(s,x,t,y) \leq L(t-s)^{-d/2}\exp\{-\mu|y-x|^2/(t-s)\},$$

$$|\nabla_x g(s,x,t,y)| \leq L(t-s)^{-(d+1)/2}\exp\{-\mu|y-x|^2/(t-s)\}.$$

As a consequence of these calculations we have the following statement.

Theorem 2. *If a given $\mathbb{R}^d$-valued function $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ is measurable and satisfies inequality (5) for some number $p>d+2$ and a given $\mathcal{L}^+(\mathbb{R}^d)$-valued function $(b(s,x))_{(s,x)\in\mathcal{D}_T}$ satisfies the conditions (i) – (ii) of Lesson 3, then the second equation in (4) has a solution*

$$G(s,x,t,y)=\sum_{k=0}^{\infty}G_k(s,x,t,y) \quad (7)$$

that belongs to the class $H_T(N,\mu,\frac{d}{2})$ with some constants $N>0$ and $\mu>0$. Moreover, in that class the solution constructed is unique.

Remark. As for the first equation in (4), it should be first transformed into an integral equation for the function $G'_x(s,x,t,y),\ 0\leq s<t\leq T,\ x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$ and then be solved by the method of successive approximations in a way similar to that for proving Theorem 2. After that the function G'_x constructed as the sum of a series like (7) should be substituted into the first equation in (4) in order to obtain the solution of it.

Exercise 4.3.A. Make sure that the solution of each equation in (4) coincides with the solution of the other one.

We now have to answer the following questions:

(a) is it true or not that the function G in Theorem 2 can serve as transition probability density of a Markov process in $\mathbb{R}^d$?

(b) if "yes", is that process a diffusion one in the sense of Kolmogorov?

To answer these questions, we show that the solution G of each one of equation (4) can be approximated with classical fundamental solutions of equation (3).

For fixed $p>d+2$, let $\mathfrak{A}_p$ be a class of functions $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ with their values in $\mathbb{R}^d$ such that $\sup_{a\in\mathfrak{A}_p}\|a\|_{p,T}<\infty$. For $k\in\{1,2\}$, let a_k be a function from $\mathfrak{A}_p$. Denote by G_k for $k\in\{1,2\}$ the solution of each one of equations (4) constructed in Theorem 2 for the function $a_k, k=1,2$.

Lemma 2. *The function $G_1(s,x,t,y)-G_2(s,x,t,y),\ 0\leq s<t\leq T,\ x\in\mathbb{R}^d, y\in\mathbb{R}^d$ belongs to the class $H_T(C\|a_1-a_2\|_{p,T},\mu,\frac{d}{2}-\gamma)$, where the constant C depends only on T,d,p,μ and $\sup_{a\in\mathfrak{A}_p}\|a\|_{p,T}$, the constant $\gamma=\frac{p-d-2}{2p}$ is as above and the constant μ is such as in Theorem 1 of Lesson 3.*

Fix a number $p>d+2$ and let an $\mathbb{R}^d$-valued function $(a(s,x))_{(s,x)\in\mathcal{D}_T}$ be such that $\|a\|_{p,T}<+\infty$. Some sequence of functions $(a_n(s,x))_{s,x\in\mathcal{D}},\ n=1,2,\dots,$ can be chosen such that

$\alpha)$ $\sup_{n\geq1}\|a_n\|_{p,T}<\infty$; $\beta)$ $\|a-a_n\|_{p,T}\to0$, as $n\to\infty$; $\gamma)$ for fixed $n\geq1$, the function a_n satisfies the condition (iii) of Lesson 3. Denote by $G(s,x,t,y)$ and $G_n(s,x,t,y)$ for $0\leq s<t\leq T,\ x\in\mathbb{R}^d, y\in\mathbb{R}^d$ and $n=1,2,\dots$ the functions constructed

in Theorem 2 for the functions a and a_n, respectively. In accordance with Lemma 2, we have the inequality

$$|G(s,x,t,y)-G_n(s,x,t,y)|\le C\|a_n-a\|_{p,T}(t-s)^{-\frac{d}{2}+\gamma}\exp\{-\mu\frac{|y-x|^2}{t-s}\}$$

valid for all $n\ge 1$, $0\le s<t\le T$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$, where C is some positive constant. Taking into account that G_n is transition probability density of a classical diffusion process, we arrive at the following statement.

Theorem 3. *Let the conditions of Theorem 2 be fulfilled. Then the function $G(s,x,t,y)$, $0\le s<t\le T$, $x\in\mathbb{R}^d, y\in\mathbb{R}^d$, defined by (7) can serve as transition probability density of a continuous Markov process in $\mathbb{R}^d$ satisfying the following equalities*

$$\int_{\mathbb{R}^d}(y-x,\theta)G(s,x,t,y)dy=\int_s^t d\tau\int_{\mathbb{R}^d}(a(\tau,y),\theta)G(s,x,\tau,y)dy, \tag{8}$$

$$\int_{\mathbb{R}^d}(y-x,\theta)^2G(s,x,t,y)dy=\int_s^t d\tau\int_{\mathbb{R}^d}(b(\tau,y)\theta,\theta)G(s,x,\tau,y)dy+ \tag{9}$$

$$+2\int_s^t d\tau\int_{\mathbb{R}^d}(a(\tau,y),\theta)(y-x,\theta)G(s,x,\tau,y)dy$$

valid for all $0\le s<t\le T$, $x\in\mathbb{R}^d$ and $\theta\in\mathbb{R}^d$.

Exercise 4.3.B. Make sure that for any $x\in\mathbb{R}^d$ and $s\in[0,T)$

$$\lim_{t\downarrow s}\frac{1}{t-s}\int_s^t d\tau\int_{\mathbb{R}^d}(a(\tau,y),\theta)(y-x,\theta)G(s,x,\tau,y)dy=0,\theta\in\mathbb{R}^d.$$

As a consequence, we have the relation

$$\lim_{t\downarrow s}\frac{1}{t-s}\int_{\mathbb{R}^d}(y-x,\theta)^2G(s,x,t,y)dy=(b(s,x)\theta,\theta)$$

held true for all $s\in[0,T)$ and $x\in\mathbb{R}^d$.

It is now evident that the relation14

$$\lim_{t\downarrow s}\frac{1}{t-s}\int_{\mathbb{R}^d}(y-x,\theta)G(s,x,t,y)dy=(a(s,x),\theta),s\in[0,T],x\in\mathbb{R}^d,\theta\in\mathbb{R}^d,$$

can be quaranteed in the case of *continuous* function a, as follows from (8).

Exercise 4.3.C. Prove that relation (8) implies the following one

$$\lim_{\Delta s\downarrow 0}\int_0^T\int_{\mathbb{R}^d}\varphi(s,x)\Big[\frac{1}{\Delta s}\int_{\mathbb{R}^d}(y-x,\theta)G(s,x,s+\Delta s,y)dy\Big]dsdx=$$

$$=\int_0^T\int_{\mathbb{R}^d}\varphi(s,x)(a(s,x),\theta)dsdx \tag{10}$$

valid for all real-valued functions φ being continuous and compactly supported on $(0,T)\times\mathbb{R}^d$.

We have thus arrived at the conclusion: a Markov process in $\mathbb{R}^d$ with its transition probability density G constructed in Theorem 2, in general, is not a dffusion process in Kolmogorov's sense. The drift vector of this process exists in some generalized sense only, as relation (10) shows.

Remark. It is not difficult to observe that in all the arguments of this section, one can put $p=+\infty$. So, these results remain to be true for the function $(a(s,x))_{(s,x)\in\mathbb{R}^d}$ being measurable and bounded.

4.4. Homogeneous processes. In the case where the local characteristics of a diffusion process do not depend on time, the results of the previous section can be obviously conformed to a homogeneous situation.

In particular, let an $\mathcal{L}^+(\mathbb{R}^d)$-valued function $(b(x))_{x\in\mathbb{R}^d}$ be given and let it satisfy the conditions (i)–(ii) of Lesson 3. Denote by $g(t,x,y)$ for $t>0, x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$ transition probability density of a diffusion process in $\mathbb{R}^d$ whose local characteristics are given by the functions $(a_0(x))_{x\in\mathbb{R}^d}$ and $(b(x))_{x\in\mathbb{R}^d}$, where $a_0(x)\equiv 0$. Let an $\mathbb{R}^d$-valued function $(a(x))_{x\in\mathbb{R}^d}$ be now given. In order to construct transition probability density $G(t,x,y), t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d$, of a diffusion process in $\mathbb{R}^d$ with its local characteristics given by the functions $(a(x))_{x\in\mathbb{R}^d}$ and $(b(x))_{x\in\mathbb{R}^d}$, we should consider the following pair of equations

$$G(t,x,y)=g(t,x,y)+\int_0^t d\tau\int_{\mathbb{R}^d} g(\tau,x,z)(a(z),G'_z(t-\tau,z,y))dz, \tag{4^0}$$

$$G(t,x,y)=g(t,x,y)+\int_0^t d\tau\int_{\mathbb{R}^d} G(\tau,x,z)(a(z),g'_z(t-\tau,z,y))dz.$$

It turns out that there exists a solution to each one of these equations under the following assumptions on a given $\mathbb{R}^d$ -valued function $(a(x))_{x\in\mathbb{R}^d}$

$$\|a\|_p=\Big(\int_{\mathbb{R}^d}|a(x)|^p dx\Big)^{1/p}<\infty \tag{5^0}$$

for some $p>d$. More precisely, the following assertion has summarized the homogeneous versions of the results expounded in the previous section.

Theorem 4. *Let an $\mathcal{L}^+(\mathbb{R}^d)$-valued function $(b(x))_{x\in\mathbb{R}^d}$ be given and let it satisfy the conditions (i)-(ii) of Lesson 3. If a given $\mathbb{R}^d$-valued function $(a(x))_{x\in\mathbb{R}^d}$ satisfies the condition (5^0) for some $p\in(d,+\infty]$, then there exists the unique solution $G(t,x,y)$, $t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d$, to each one of equations (4^0) satisfying the inequalities*

$$G(t,x,y)\le Kt^{-\frac{d}{2}}\exp\{-\mu\frac{|y-x|^2}{t-s}\},$$

$$|G'_x(t,x,y)|\le Kt^{-\frac{d+1}{2}}\exp\{-\mu\frac{|y-x|^2}{t-s}\}$$

in each domain $(t,x,y)\in(0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ for $T<\infty$ (the constant K may depend on T). That solution can serve as transition probability density of a homogeneous Markov process in $\mathbb{R}^d$ possessing the following properties

$$\int_{\mathbb{R}^d}(y-x,\theta)G(t,x,y)dy=\int_0^t d\tau\int_{\mathbb{R}^d}(a(y),\theta)G(\tau,x,y)dy,$$

$$\int_{\mathbb{R}^d}(y-x,\theta)^2G(t,x,y)dy=\int_0^t d\tau\int_{\mathbb{R}^d}(b(y)\theta,\theta)G(\tau,x,y)dy+$$
$$+2\int_0^t d\tau\int_{\mathbb{R}^d}(a(y),\theta)(y-x,\theta)G(\tau,x,y)dy,$$

where $t>0, x\in\mathbb{R}^d$ and $\theta\in\mathbb{R}^d$.

Clearly, this process is a diffusion one in the Kolmogorov sense if $(a(x))_{x\in\mathbb{R}^d}$ is a continuous function. In the general case (that is, only condition (5^0) is supposed to be fulfilled), the relation

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^d}\varphi(x)\Big[\frac{1}{t}\int_{\mathbb{R}^d}(y-x,\theta)G(t,x,y)dy\Big]dx=\int_{\mathbb{R}^d}a(x)\varphi(x)dx$$

holds true for any continuous compactly supported function $(\varphi(x))_{x\in\mathbb{R}^d}$.

Exercise 4.4.A. Make sure that under the conditions of Theorem 4, the method of successive approximations is applicable to each one of equations (4^0); construct the function G in a way like the series (7).

Exercise 4.4.B. Let $(x(t), \mathcal{M}_t, \mathbb{P}_x^{a,b})$ be a continuous homogeneous Markov process in $\mathbb{R}^d$ whose transition probability density $G(t,x,y), t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d$, is determined by equations (4^0). Suppose that the space of elementary events of this process coincides with all continuous $\mathbb{R}^d$-valued functions (see Lesson 1). Denote by $(x(t), \mathcal{M}_t, \mathbb{P}_x^{0,b})$ a continuous homogeneous Markov process with its transition probability density given by the function $g(t,x,y), t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d$ and let its space of elementary events be the same as above.

Prove that for all $x\in\mathbb{R}^d$ and $T<+\infty$, the restrictions of the measures $\mathbb{P}_x^{a,b}$ and $\mathbb{P}_x^{0,b}$ on the σ-algebra $\mathcal{M}_T$ are equivalent in the cases: (a) $d\geq 2$ and $\|a\|_p<\infty$ for some $p>d$; (b) $d=1$ and $\|a\|_p<\infty$ for some $p\geq 2$. Those restrictions are not equivalent in the case of $\|a\|_p<\infty$ for some $p\in(1,2)$, but $\int_{-N}^N |a(x)|^2dx=+\infty$ for some $N>0$.

4.5. Comments and references. The results of this section are expounded in details in the book [8].

Lesson 5. Diffusion processes in a medium with membranes located on given surfaces.

5.1. Introduction. In this lesson we show how to construct a continuous homogeneous Markov process in $\mathbb{R}^d$ with its diffusion operator being identically equal to an identity operator in $\mathbb{R}^d$ and its drift vector given by a function of the form $(\nu(x)q(x)\delta_S(x))_{x\in\mathbb{R}^d}$, where S is a given hypersurface in $\mathbb{R}^d$, $(\delta_S(x))_{x\in\mathbb{R}^d}$ is a generalized function whose action on a test function $(\varphi(x))_{x\in\mathbb{R}^d}$ is defined by $\langle\delta_S,\varphi\rangle=\int_S\varphi(x)d\sigma$ (this is a surface integral), $\nu(x)$ for $x\in S$ is a unit vector being orthogonal to S at the point x and $(q(x))_{x\in S}$ is a given continuous function with its values in the interval $[-1,1]$. It is clear that such a process cannot be a diffusion one in the Kolmogorov sense. Nevertheless, its local characteristics do exist in some generalized sense. In Lessons 1 and 2 we have already delt with some examples of the kind: in the case of $d=1$, S is reduced to the point ($S=\{0\}$) and the corresponding process is called skew Brownian motion; if $d\geq 2$ and S is a hyperplane in $\mathbb{R}^d$, then the corresponding process is called multidimensional Brownian motion with a membrane located on that given hyperplane, and its transition probability density is given by an explicit formula.

5.2. Single–layer potentials. Suppose that S is a closed bounded hypersurface separating $\mathbb{R}^d (d\geq 2)$ into two open parts: the interior $\mathcal{D}_i$ and the exterior $\mathcal{D}_e$, so that $\mathbb{R}^d=\mathcal{D}_i\cup\mathcal{D}_e\cup S$. Assume that there exists the unique tangent plane at each point $x\in S$. Let $\nu(x)$ be the unit outer normal vector to S at x. For $x\in S$ we construct a so-called local system of coordinates, i.e., a rectangular system of coordinates $(y^1,y^2,\dots,y^d)$ with the origin at x and with the direction of the axis y^d along $\nu(x)$. It is assumed that for some $r_0>0$ and each $x\in S$ the piece of surface $S_{r_0}(x)=S\cap B_{r_0}(x)$ can be given in the local system of coordinates (with the origin at x) by an equation

$$y^d=F(y^1,y^2,\dots,y^{d-1}),$$

where F is a single-valued function. Recall that S is called a surface of class $H^{1+\delta}$ for some $\delta\in(0,1]$ if for every $x\in S$ the corresponding function F has in the domain $\sum_{j=1}^{d-1}(y^j)^2\leq\frac{r_0^2}{4}$ continuous partial derivatives $\frac{\partial F}{\partial y^k}$, $k=1,2,\dots,d-1$, satisfying in this domain a Hölder condition with exponent δ and a constant independent of x. It will be assumed below that the closed hypersurface S belongs to the class $H^{1+\delta}$ for some $\delta \in (0,1]$.

To simplify the further exposition, we assume that a diffusion process to be perturbed is Brownian motion in $\mathbb{R}^d$, that is, a continuous homogeneous Markov process in $\mathbb{R}^d$ with its transition probability density given by

$$g(t,x,y) = (2\pi t)^{-d/2} \exp\{-\frac{|y-x|^2}{2t}\},\ t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d.$$

For a given real-valued measurable function $(\varphi(t,x))_{(t,x)\in(0,T]\times S}$ satisfying the inequality

$$|\varphi(t,x)| \le K_T t^{\beta},\ t\in(0,T],\ x\in S, \tag{1}$$

with some constants $\beta > -1$ and $K_T > 0$ (K_T is used to denote various constants), define a function $\Phi(t,x)$ for $(t,x)\in\mathcal{D}_T$ by the integrals

$$\Phi(t,x) = \int_0^t d\tau \int_S g(t-\tau,x,y)\varphi(\tau,y)d\sigma_y, \tag{2}$$

where the inner integral is a surface one. The fact that the integrals in (2) are well-defined is a consequence of the following evident estimation

$$\int_S g(t,x,y)d\sigma_y \le K_T t^{-1/2},\ (t,x)\in(0,T]\times\mathbb{R}^d. \tag{3}$$

This inequality allows us to assert that the function $(\Phi(t,x))_{(t,x)\in(0,T]\times\mathbb{R}^d}$ is continuous and satisfies the inequality

$$|\Phi(t,x)| \le K_T t^{\beta+1/2},\ t\in(0,T],\ x\in\mathbb{R}^d.$$

Moreover, one can easily observe that in the domain $t\in(0,T], x\notin S$, the function Φ satisfies the heat equation

$$\frac{\partial\Phi}{\partial t} = \frac{1}{2}\Delta\Phi.$$

Now, for $x\notin S$ and $x_0\in S$ ($t>0$ is fixed), the derivative of the function $\Phi(t,x)$ in the direction $\nu(x_0)$ is well-defined and can be written as follows

$$\frac{\partial\Phi(t,x)}{\partial\nu(x_0)} = \int_0^t d\tau \int_S \frac{(y-x,\nu(x_0))}{t-\tau} g(t-\tau,x,y)\varphi(\tau,y)d\sigma_y.$$

The behavior of this derivative, as $x\to x_0$, is described by the following theorem on the normal derivative of a single-layer potential. This theorem is one of the most beautiful theorems of classical analysis.

It turns out that the limit of this derivative depends on the way the point x is approaching the point $x_0\in S$. If a given function $(h(s,x))_{(s,x)\in\mathcal{D}_T}$ has the limit, as $x\to x_0$ along an arbitrary curve lying in some finite closed cone $\mathcal{K}$ in $\mathbb{R}^d$ with vertex at x_0 such that $\mathcal{K}\subset\mathcal{D}_i\cup\{x_0\}$, then we say that the function $(h(s,x))_{(s,x)\in\mathcal{D}_T}$ has a non-tangent inner limit at the point $x_0\in S$, and it is denoted by $h(s,x_0-)$. A non-tangent outer limit is defined analogously, but this time the inclusion $\mathcal{K}\subset\mathcal{D}_e\cup\{x_0\}$ must be held, and $h(s,x_0+)$ is the designation for this limit.

One more remark should be made before formulating the theorem. If the hypersurface S belongs to the class $H^{1+\delta}$, then for $x_0\in S$ and $y\in S\cap B_{r_0/2}(x_0)$, the inequality $|(y-x_0,\nu(x_0))| \le \text{const}\cdot|y-x_0|^{1+\delta}$ holds true. As a consequence, we have the following estimation

$$\int_0^t d\tau\int_S \Big|\frac{\partial g(t-\tau,x_0,y)}{\partial\nu(x_0)}\cdot\varphi(\tau,y)\Big|dy \le \text{const}\int_0^t (t-\tau)^{-1+\frac{\delta}{2}}\tau^{\beta}d\tau = \text{const}\cdot t^{\beta+\frac{\delta}{2}} \tag{4}$$

valid for $t \in (0, T]$, $x_0 \in S$ and any measurable function $(\varphi(\tau, y))_{(\tau,y)\in(0,T]\times S}$ satisfying inequality (1). The function

$$\int_0^t d\tau \int_S \frac{\partial g(t-\tau, x_0, y)}{\partial \nu(x_0)} \varphi(\tau, y) dy, \ t \in (0, T], x_0 \in S,$$

is called the direct value of the normal derivative $\frac{\partial \Phi(t,x)}{\partial \nu(x_0)}$ at the point $x = x_0$. The fact that this function is well-defined follows from (4).

Theorem 1. *If a closed bounded hypersurface S belongs to the class $H^{1+\delta}$, and the function $(\varphi(t, x))_{(t,x)\in(0,T]\times S}$ is continuous and satisfies inequality (1), then for $t \in (0, T]$ and $x_0 \in S$*

$$\frac{\partial \Phi}{\partial \nu(x_0)}(t, x_0\pm) = \mp\varphi(t, x_0) + \int_0^t d\tau \int_S \frac{\partial g(t-\tau, x_0, y)}{\partial \nu(x_0)} \varphi(\tau, y) d\sigma_y.$$

This statement is known as the theorem on the jump of the normal derivative of a single-layer potential.

The version of this theorem for S being a hyperplane in $\mathbb{R}^d (d \geq 2)$ is particularly simple: since

$$\frac{\partial g(t, x, y)}{\partial \nu} = \frac{(y - x, \nu)}{t} g(t, x, y) = 0, \ t > 0, x \in S, y \in S,$$

where ν is a unit vector normal to S, we have the relations

$$\lim_{x\to x_0\pm} \int_0^t d\tau \int_S \frac{\partial g(t-\tau, x, y)}{\partial \nu_x} \varphi(\tau, y) d\sigma_y = \mp\varphi(t, x_0), t > 0, x_0 \in S,$$

valid for any continuous function $\varphi(t, y), t > 0, y \in S$, satisfying condition (1) (index x at the letter ν indicates the argument with respect to which the derivative $\frac{\partial}{\partial \nu}$ is taken).

If $d = 1$ ($S = \{0\}$), then a simple-layer potential can be written as follows

$$\Phi(t, x) = \int_0^t g(t - \tau, x, 0)\varphi(\tau) d\tau, \ t > 0, x \in \mathbb{R}^1.$$

In this case the relations

$$\lim_{x\to 0\pm} \frac{\partial \Phi(t, x)}{\partial x} = \mp\varphi(t), t > 0,$$

hold true for any continuous function $(\varphi(\tau))_{\tau>0}$ satisfying the condition $\int_0^\varepsilon |\varphi(s)| ds < \infty$ for some (hence, for any) $\varepsilon > 0$.

5.3. The integral equations. Let a closed bounded hypersurface S in $\mathbb{R}^d$ ($d \geq 2$) be given such as in Theorem 1. Our starting point in this section is the following pair of perturbation formulae (see equations (4^0) in the previous lesson)

$$G(t, x, y) = g(t, x, y) + \int_0^t d\tau \int_{\mathbb{R}^d} g(t - \tau, x, z)(a(z), G'_z(\tau, z, y)) dz,$$

(5)

$$G(t, x, y) = g(t, x, y) + \int_0^t d\tau \int_{\mathbb{R}^d} G(t - \tau, x, z)(a(z), g'_z(\tau, z, y)) dz.$$

Recall that $g(t, x, y) = (2\pi t)^{-d/2} \exp\{-|y - x|^2/2t\}$ for $t > 0, x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ in this lesson. We are going to put here $a(x) = \nu(x)q(x)\delta_S(x), x \in \mathbb{R}^d$. One should first guess that the function $(\frac{\partial G}{\partial \nu(x_0)}(t, x, y))_{x\in\mathbb{R}^d}$ for fixed $t > 0, x_0 \in S$ and $y \in \mathbb{R}^d$ must have a jump at those points $x_0 \in S$, where $q(x_0) \neq 0$.

We have thus come to the conclusion that some sense to the product $(\psi(x)\delta_S(x))_{x\in\mathbb{R}^d}$ must be attached in the case of a function $(\psi(x))_{x\in\mathbb{R}^d}$ having non-tangent limits $\psi(x_0\pm)$ at the points $x_0 \in S$. We propose the following rule

$$\langle\delta_S, \psi\rangle = \frac{1}{2}\int_S [\psi(z+) + \psi(z-)]d\sigma.$$

According to this rule, the first equation in (5) can be rewritten as follows

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau \int_S g(t-\tau,x,z)V(\tau,z,y)q(z)d\sigma_z, \tag{6}$$

where

$$V(\tau,z,y) = \frac{1}{2}\Big[\frac{\partial G}{\partial\nu(z)}(\tau,x,y)\Big|_{x=z+} + \frac{\partial G}{\partial\nu(z)}(\tau,x,y)\Big|_{x=z-}\Big]$$

for $\tau > 0, z \in S$ and $y \in \mathbb{R}^d$.

Similar reasonings concern the second equation in (5), but this time the function $(G(t,x,y))_{y\in\mathbb{R}^d}$ for fixed $t > 0$ and $x \in \mathbb{R}^d$ must have a jump at the point $y \in S$, where $q(y) \neq 0$. Hence, putting

$$\widetilde{V}(t,x,z) = \frac{1}{2}[G(t,x,z+) + G(t,x,z-)]$$

for $t > 0, x \in \mathbb{R}^d$ and $z \in S$, we get

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau \int_S \widetilde{V}(t-\tau,x,z)\frac{\partial g}{\partial\nu(z)}(\tau,z,y)q(z)d\sigma_z. \tag{7}$$

Equation (6) shows that the function G is completely determined by the function V. On the other hand, the integrals on the right-hand side of (6) are nothing else but a single-layer potential. Applying Theorem 1 to (6), we get for $t > 0, x \in S$ and $y \in \mathbb{R}^d$

$$\frac{\partial G}{\partial\nu(x)}(t,x\pm,y) = \frac{\partial g}{\partial\nu(x)}(t,x,y) \mp q(x)V(t,x,y)+$$
$$+\int_0^t d\tau \int_S \frac{\partial g}{\partial\nu(x)}(t-\tau,x,z)V(\tau,z,y)q(z)d\sigma_z. \tag{8}$$

These equalities imply the integral equation for the function $V(t,x,y), t > 0, x \in S, y \in \mathbb{R}^d$,

$$V(t,x,y) = \frac{\partial g(t,x,y)}{\partial\nu(x)} + \int_0^t d\tau \int_S \frac{\partial g(t-\tau,x,z)}{\partial\nu(x)}V(\tau,z,y)q(z)d\sigma_z. \tag{9}$$

On the other hand, the relations

$$\frac{\partial G}{\partial\nu(x)}(t,x\pm,y) = (1\mp q(x))V(t,x,y) \tag{10}$$

valid for $t > 0, x \in S$ and $y \in \mathbb{R}^d$, are also simple consequences of (8).

Now, taking into account the relations

$$\frac{\partial g}{\partial\nu(z)}(\tau,z,y) = (\nu(z), g'_z(\tau,z,y)) = -(\nu(z), g'_y(\tau,z,y))$$

valid for $\tau > 0, z \in S$ and $y \in \mathbb{R}^d$, and applying Theorem 1 once again, we get from (7) the equalities

$$G(t,x,y\pm) = g(t,x,y) \pm q(y)\widetilde{V}(t,x,y)+$$
$$+\int_0^t d\tau \int_S \widetilde{V}(t-\tau,x,z)\frac{\partial g}{\partial\nu(z)}(\tau,z,y)q(z)d\sigma_z$$

valid for $t > 0, x \in \mathbb{R}^d$ and $y \in S$. As a consequence, we have the integral equation for $\widetilde{V}(t,x,y), t > 0, x \in \mathbb{R}^d, y \in S$

$$\widetilde{V}(t,x,y) = g(t,x,y) + \int_0^t d\tau \int_S \widetilde{V}(t-\tau,x,z) \frac{\partial g}{\partial \nu(z)}(\tau,z,y) q(z) d\sigma_z \tag{11}$$

and also the relations

$$G(t,x,y\pm) = (1 \pm q(y))\widetilde{V}(t,x,y), \ t > 0, x \in \mathbb{R}^d, y \in S. \tag{12}$$

Our next step consists in constructing solutions to equations (9) and (11).

5.4. Solving equations (9) and (11). Denote by $Q(t,x,y)$ for $t > 0, x \in S$, and $y \in S$ the restriction of the function $\dfrac{\partial g(t,x,y)}{\partial \nu(x)}$ on the set $(0,+\infty) \times S \times S$. Making use of the inequality $|(y-x,\nu(x))| \leq \text{const}\, |y-x|^{1+\delta}$ valid for all $x \in S$ and $y \in S$ with some constant depending on S only (recall that the bounded closed surface S belongs to the class $H^{1+\delta}$ with some $\delta \in (0,1]$), we can estimate the function Q as follows

$$|Q(t,x,y)| \leq \text{const} \cdot t^{-\frac{d+2}{2}} |y-x|^{1+\delta} \exp\{-|y-x|^2/2t\} \leq \frac{L}{t^{\varkappa}|y-x|^{d+1-2\varkappa-\delta}}, \tag{13}$$

where $\varkappa$ can be arbitrarily chosen from the interval $(1-\frac{\delta}{2}, 1)$ and L is some positive constant depending on S and $\varkappa$. Such a choice of $\varkappa$ implies the inequalities $d+1-2\varkappa-\delta < d-1,\ 2\varkappa+\delta-2 > 0$. We put $\rho = d+1-2\varkappa-\delta,\ \gamma = 2\varkappa+\delta-2$ and $\sigma = 1-\varkappa$; then $\gamma > 0$ and $\sigma > 0$.

Let us now define a sequence of functions $(Q^{(k)})_{k\geq 1}$ given on the set $(0,+\infty) \times S \times S$ by setting $Q^{(1)} = Q$ and

$$Q^{(k+1)}(t,x,y) = \int_0^t d\tau \int_S Q(\tau,x,z) Q^{(k)}(t-\tau,z,y) q(z) d\sigma_z,$$

where $(q(z))_{z\in S}$ is a fixed real-valued continuous function (we will use the notation $\|q\| = \max_{z\in S} |q(z)|$). It is evident that

$$Q^{(k+1)}(t,x,y) = \int_0^t d\tau \int_S Q^{(k)}(\tau,x,z) Q(t-\tau,z,y) q(z) d\sigma_z$$

for all $t > 0, x \in S$, and $y \in S$. Making use of estimation (13) and Lemma 2 from [12] (see Ch.V. §2), we get for $(t,x,y) \in (0,T] \times S \times S$

$$|Q^{(2)}(t,x,y)| \leq \frac{\text{const}}{t^{\varkappa-\sigma}|y-x|^{\rho-\gamma}},\ |Q^{(3)}(t,x,y)| \leq \frac{\text{const}}{t^{\varkappa-2\sigma}|y-x|^{\rho-2\gamma}}.$$

Therefore, an integer k_0 exists such that $|Q^{(k_0)}(t,x,y)| \leq C_T$ for all $(t,x,y) \in (0,T] \times S \times S$ with some constant $C_T > 0$. Then by induction on n, we arrive at the estimation

$$|Q^{(n+k_0)}(t,x,y)| \leq C_T^n \frac{t^{n-\varkappa}}{(1-\varkappa)(2-\varkappa)\dots(n-\varkappa)}$$

valid for all $(t,x,y) \in [0,T] \times S \times S$ and $n = 1,2,\dots$. We have thus proved the following assertion.

Lemma 1. *The series*

$$R(t,x,y) = \sum_{k=1}^{\infty} Q^{(k)}(t,x,y)$$

is convergent uniformly in $x \in S$ and $y \in S$ and locally uniformly in $t > 0$. The kernel R is continuous in the arguments $t > 0, x \in S, y \in S, y \neq x$, and satisfies the inequality

$$|R(t,x,y)| \leq \frac{K_T}{t^{\varkappa_1}|y-x|^{\varkappa_2}} \tag{14}$$

in any domain of the form $(0,T]\times S\times S$ *with some constants* $\varkappa_1\in(0,1)$ *and* $\varkappa_2\in(0,d-1)$. *In addition, this kernel is the solution to each one of the following pair of integral equations* $(t>0, x\in S, y\in S)$

$$R(t,x,y)=Q(t,x,y)+\int_0^t d\tau\int_S R(\tau,x,z)Q(t-\tau,z,y)q(z)d\sigma_z, \tag{15}$$

$$R(t,x,y)=Q(t,x,y)+\int_0^t d\tau\int_S Q(\tau,x,z)R(t-\tau,z,y)q(z)d\sigma_z.$$

Finally, each equation in (15) has no more than one solution satisfying estimation (14).

Corollary. The solution to equation (9) can be given by

$$V(t,x,y)=\frac{\partial g(t,x,y)}{\partial\nu(x)}+\int_0^t d\tau\int_S R(\tau,x,z)\frac{\partial g(t-\tau,z,y)}{\partial\nu(z)}q(z)d\sigma_z \tag{16}$$

for $t>0, x\in S$ and $y\in\mathbb{R}^d$; and the solution to equation (11) can be written as follows

$$\widetilde{V}(t,x,y)=g(t,x,y)+\int_0^t d\tau\int_S g(\tau,x,z)R(t-\tau,z,y)q(z)d\sigma_z \tag{17}$$

for $t>0, x\in\mathbb{R}^d$, and $y\in S$.

Exercise 5.4.A. Make sure that substituting (16) into (6) leads us to the function G being the same as the result of substituting (17) into (7).

We have thus obtained two different representations for the function G. Our nearest aim is to show that under some additional assumptions on S, the function G is indeed transition probability density of the process desired.

5.5. Properties of the function G. The assertions of this section are proposed to the reader as exercises provided by some hints.

Denote by $\mathbb{B}(\mathbb{R}^d)$ the Banach space of all real-valued measurable bounded functions $(\varphi(x))_{x\in\mathbb{R}^d}$ with the norm $\|\varphi\|=\sup_{x\in\mathbb{R}^d}|\varphi(x)|$. The designation $\mathbb{C}(\mathbb{R}^d)$ is used for the Banach space consisting of all continuous functions from $\mathbb{B}(\mathbb{R}^d)$ with the same norm.

Notice that equality (16) implies the following estimation

$$\Big|\int_{\mathbb{R}^d} V(t,x,y)\varphi(y)dy\Big|\le K_T\|\varphi\|t^{-1/2} \tag{18}$$

valid for all $t\in(0,T], x\in S$ and $\varphi\in\mathbb{B}(\mathbb{R}^d)$. So, if we put

$$u(t,x,\varphi)=\int_{\mathbb{R}^d} G(t,x,y)\varphi(y)dy,\ t>0, x\in\mathbb{R}^d,\ \varphi\in\mathbb{B}(\mathbb{R}^d),$$

then for any $T>0$, there exists a constant $C_T>0$ such that the inequality $|u(t,x,\varphi)|\le C_T\|\varphi\|$ holds true for all $t\in(0,T], x\in\mathbb{R}^d$ and $\varphi\in\mathbb{B}(\mathbb{R}^d)$.

Exercise 5.5.A. Verify that the function $u(t,x,\varphi)$ defined above for $t>0, x\in\mathbb{R}^d$ and $\varphi\in\mathbb{B}(\mathbb{R}^d)$ is a solution to the heat equation in any domain of the kind $(t,x)\in(0,T]\times(\mathcal{D}_i\cup\mathcal{D}_e)$ for all $T>0$. In the case of $\varphi\in\mathbb{C}(\mathbb{R}^d)$, the initial condition $u(0+,x,\varphi)=\varphi(x)$ is fulfilled at any point $x\in\mathbb{R}^d$.

Hints. The first statement follows from the equality

$$u(t,x,\varphi)=\int_{\mathbb{R}^d} g(t,x,y)\varphi(y)dy+\int_0^t d\tau\int_S g(t-\tau,x,z)q(z)\Big[\int_{\mathbb{R}^d} V(\tau,z,y)\varphi(y)dy\Big]d\sigma_z.$$

For proving the second one, make use of the equality

$$\int_{\mathbb{R}^d} V(t,x,y)dy=0,\ t>0, x\in S.$$

Exercise 5.5.B. Verify that for all $t_1 > 0, t_2 > 0, x \in \mathbb{R}^d$ and $\varphi \in \mathbb{B}(\mathbb{R}^d)$, the following relation

$$u(t_1 + t_2, x, \varphi) = u(t_1, x, u(t_2, \cdot, \varphi)) \tag{19}$$

holds true.

Hint. Establish first the relation

$$\int_{\mathbb{R}^d} V(s+t, x, y)\varphi(y)dy = \int_{\mathbb{R}^d} V(t, x, z)u(s, z, \varphi)dz$$

valid for all $s > 0, t > 0, x \in S$ and $\varphi \in \mathbb{B}(\mathbb{R}^d)$.

As follows from (12), the function G in its third argument is discontinuous. Let us believe that $G(t, x, y) = g(t, x, y)$ for all $t > 0, x \in \mathbb{R}^d$ and $y \in S$. Then a very simple consequence of (19) is the relation

$$G(s+t, x, y) = \int_{\mathbb{R}^d} G(s, x, z)G(t, z, y)dz$$

valid for all $s > 0, t > 0, x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$. In other words, the function G satisfies the Kolmogorov–Chapman equation.

Exercise 5.5.C. Make sure that if $\varphi \in \mathbb{B}(\mathbb{R}^d)$ and $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^d$, then $u(t, x, \varphi) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}^d$ (under some additional requirements on the surface S, see below).

Hints. Show first that if $\varphi_n \in \mathbb{B}(\mathbb{R}^d)$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}^d$ and $\sup_n \|\varphi_n\| < \infty$, then $\lim_{n\to\infty} u(t, x, \varphi_n) = u(t, x, \varphi)$. Consequently, it suffices to prove that for any compactly supported function $\varphi \in \mathbb{B}(\mathbb{R}^d)$ being smooth enough, we have $u(t, x, \varphi) \geq 0$ for all $x \in \mathbb{R}^d$ if only $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^d$.

Now, let φ be a function on $\mathbb{R}^d$ compactly supported, twice continuously differentiable and bounded along with its derivatives. Then the function $u(t, x, \varphi), t > 0, x \in \mathbb{R}^d$, possesses the following properties

(i) it satisfies the heat equation in the domain $(0, +\infty) \times (\mathcal{D}_i \cup \mathcal{D}_e)$;

(ii) it satisfies the initial condition $u(0+, x, \varphi) = \varphi(x), x \in \mathbb{R}^d$;

(iii) the following relations

$$\frac{\partial u(t, x\pm, \varphi)}{\partial \nu(x)} = (1 \mp q(x)) \int_{\mathbb{R}^d} V(t, x, y)\varphi(y)dy \tag{20}$$

hold true for all $t > 0$ and $x \in S$ (see (10)).

Let $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^d$. If for some $T > 0$, we have $\inf_{(t,x)\in[0,T]\times\mathbb{R}^d} u(t, x, \varphi) = \gamma < 0$, then there exists $t_0 \in (0, T]$ and $x_0 \in \mathbb{R}^d$ such that $u(t_0, x_0, \varphi) = \gamma$. It is not difficult to comprehend that $x_0 \in S$. Therefore the inequalities

$$\frac{\partial u(t_0, x_0+, \varphi)}{\partial \nu(x_0)} \geq 0 \text{ and } \frac{\partial u(t_0, x_0-, \varphi)}{\partial \nu(x_0)} \leq 0 \tag{21}$$

are fulfilled.

We now show that under some additional assumptions on surface S, any equality is not allowed in these inequalities.

Definition. We say that a point $x \in S$ has the property of inner sphericity if there exists a closed ball $B \subset \mathcal{D}_i \cup \{x\}$ such that $x \in \partial B$. The property of outer sphericity is defined similarly.

The proof of the following assertion is based on Theorem 14 in [12], Chapter II, §5, in which an essential role is played by the assumption about the sphericity property of the points of S.

Lemma 2. *Assume that the surface S belongs to the class $H^{1+\delta}$ and, moreover, each point $x \in S$ has the property of both inner and outer sphericity. Then instead of (21),* *the strict inequalities*

$$\frac{\partial u(t_0, x_0+, \varphi)}{\partial \nu(x_0)} > 0 \text{ and } \frac{\partial u(t_0, x_0-, \varphi)}{\partial \nu(x_0)} < 0 \tag{22}$$

hold true.

Let us return to Exercise 5.5.C. Recall that the continuous function $(q(x))_{x\in S}$ was supposed to take on its values from the interval $[-1, 1]$. But then relations (20) contradict the ones in (22). This means that our supposition that $\gamma < 0$ is not true. In other words, if $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^d$, then also $u(t, x, \varphi) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}^d$.

As a consequence of Exercises 5.5.A and 5.5.C, we have the following assertion.

Corollary. Under the assumptions of Lemma 2, the following inequality

$$|u(t, x, \varphi)| \leq \|\varphi\| \tag{23}$$

holds true for all $t > 0, x \in \mathbb{R}^d$ and $\varphi \in \mathbb{B}(\mathbb{R}^d)$.

Taking into account now that $\int_{\mathbb{R}^d} G(t, x, y)dy \equiv 1$ (this follows from the equality $\int_{\mathbb{R}^d} V(t, x, y)dy = 0$ valid for all $t > 0$ and $x \in S$), we can assert that the function G can serve as transition probability density of a Markov process in $\mathbb{R}^d$. The next exercise proposes to prove that the continuity condition is fulfilled for this process.

Exercise 5.5.D. Suppose that conditions of Lemma 2 are fulfilled. Verify that the inequality

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |y - x|^4 G(t, x, y)dy \leq K_T t^2 \tag{24}$$

holds true for all $t \in (0, T]$ with some positive constant K_T finite for $T < +\infty$.

Hints. Show first that for fixed $x_0 \in \mathbb{R}^d$ and $T > 0$, the inequality

$$\int_{\mathbb{R}^d} |V(t, x, y)||y - x_0|^4 dy \leq K_T(|x - x_0|^4 + t^2)t^{-1/2}$$

is valid for all $t \in (0, T]$ and $x \in S$. Deduce from this that

$$\int_{\mathbb{R}^d} |y - x_0|^4 G(t, x_0, y)dy \leq K_T t^2 +$$

$$+K_T \int_0^t d\tau \int_S (2\pi(t-\tau))^{-d/2} \exp\Big\{-\frac{|y - x_0|^2}{2(t-\tau)}\Big\}[\tau^{3/2} + |y - x_0|^4 \tau^{-1/2}]d\sigma_y$$

and this implies (24).

The following statement summarizes the consideretions of this section.

Theorem 2. *Let a closed bounded hypersurface S in $\mathbb{R}^d$ belong to the class $H^{1+\delta}$ for some $\delta > 0$ and let every point of S possess the property of inner and outer sphericity. Then there exists a continuous homogeneous Markov process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ in $\mathbb{R}^d$ whose transition probability density is given by the function $G(t, x, y), t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d$, defined by equality (6) or (7) with a given continuous function $(q(x))_{x\in S}$ taking on its values from the interval* $[-1, 1]$.

Exercise 5.5.E. Establish the relations

$$\int_{\mathbb{R}^d} (y - x, \theta)G(t, x, y)dy = \int_0^t d\tau \int_S (\nu(z), \theta)\widetilde{V}(\tau, x, z)q(z)d\sigma_z$$

$$\int_{\mathbb{R}^d} (y - x, \theta)^2 G(t, x, y)dy = t|\theta|^2 + 2\int_0^t d\tau \int_S (\nu(z), \theta)(z - x, \theta)\widetilde{V}(\tau, x, z)q(z)d\sigma_z$$

valid for $t > 0, x \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^d$.

These relations imply the following assertion.

Theorem 3. *The Markov process in Theorem 2 is a diffusion process in the following sense: for any continuous compactly supported function* $(\varphi(x))_{x\in\mathbb{R}^d}$ *and* $\theta \in \mathbb{R}^d$, *the relations*

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^d}\varphi(x)\Big[\frac{1}{t}\int_{\mathbb{R}^d}(y-x,\theta)G(t,x,y)dy\Big]dx = \int_S(\nu(y),\theta)q(y)\varphi(y)d\sigma_y,$$

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^d}\varphi(x)\Big[\frac{1}{t}\int_{\mathbb{R}^d}(y-x,\theta)^2G(t,x,y)dy\Big]dx = |\theta|^2\int_{\mathbb{R}^d}\varphi(y)dy$$

hold true.

Section 5.6. Comments and references. The results of this lesson (and more general ones) can be found in [16], [8].

References

16. O.V.Aryasova, M.I.Portenko, *One example of a random change of time that transforms a generalized diffusion process into an ordinary one*, Theory of Stochastic Processes, **13(29)** (2007), no. 3, 12–21.

Lesson 6. Multidimensional Brownian motion with a membrane being located on a given hyperplane and acting in an oblique direction.

6.1. Introduction. Using the methods developped in the previous lessons, we now show how to construct a diffusion process in $\mathbb{R}^d$, $d\geq 2$, such that its diffusion operator coicides with an identity operator in $\mathbb{R}^d$ and its drift vector is given by the function $(N(x)\delta_S(x))_{x\in\mathbb{R}^d}$, where S is the hyperplane in $\mathbb{R}^d$ being orthogonal to a fixed unit vector $\nu\in\mathbb{R}^d$; $(N(x))_{x\in S}$ is a given $\mathbb{R}^d$-valued vector field on S; $(\delta_S(x))_{x\in\mathbb{R}^d}$ is a distribution (a generalized function) on $\mathbb{R}^d$ whose action on a test function $(\varphi(x))_{x\in\mathbb{R}^d}$ is given by $\langle\delta_S,\varphi\rangle = \int_S\varphi(x)d\sigma$ (this is a surface integral, which, in fact, is the integral over $\mathbb{R}^{d-1}$ with respect to the Lebesgue measure in $\mathbb{R}^{d-1}$). If, in particular, $N(x)=\nu q(x)$ for $x\in S$ with some continuous function $(q(x))_{x\in S}$ taking on its values from the interval $[-1,1]$, then the process of this lesson is the one described in the Examples 1.4.D and 2.4.C above. More precisely, the process of those examples is our starting point for further perturbing it by the vector field $(\alpha(x)\delta_S(x))_{x\in\mathbb{R}^d}$, where $\alpha(x)=N(x)-\nu q(x)$ for $x\in S$.

6.2. An integro-differential equation for a diffusion perturbed. So, we are given by a fixed unit vector $\nu\in\mathbb{R}^d, d\geq 2$. Denote by S the subspace of $\mathbb{R}^d$ that is orthogonal to $\nu: S=\{x\in\mathbb{R}^d:(x,\nu)=0\}$. The open half-spaces $\{x\in\mathbb{R}^d:(x,\nu)>0\}$ and $\{x\in\mathbb{R}^d:(x,\nu)<0\}$ are denoted, respectively, by $\mathcal{D}_+$ and $\mathcal{D}_-$. Let continuous functions $(\alpha(x))_{x\in S}$ and $(q(x))_{x\in S}$ with their values, respectively, in S and the interval $[-1,1]$ be given. We set $N(x)=\alpha(x)+\nu q(x)$ for $x\in S$.

Denote by $g_0(t,x,y)$ for $t>0, x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$ transition probability density of Brownian motion in $\mathbb{R}^d$

$$g_0(t,x,y)=(2\pi t)^{-d/2}\exp\{-|y-x|^2/2t\}.$$

By $g(t,x,y)$ for $t>0, x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$, we denote transition probability density of the process considered in Examples 1.4.D and 2.4.C above, that is

$$g(t,x,y)=g_0(t,x,y)+\int_0^t d\tau\int_S g_0(t-\tau,x,z)\frac{(y,\nu)}{\tau}g_0(\tau,z,y)q(z)d\sigma_z. \qquad (1)$$

Notice that the second item on the right-hand side of (1) is nothing else but taken with the sign minus the normal derivative (in y) of a single-layer potential. According to Section 5.2, we have the relations

$$g(t,x,y\pm)=(1\pm q(y))g_0(t,x,y) \qquad (2)$$

valid for $t>0, x\in\mathbb{R}^d$ and $y\in S$ ($g(t,x,y+)$ means the non-tangent limit of $g(t,x,z)$, as $z\to y$ in such a way that $z\in\mathcal{D}_+$; $g(t,x,y-)$ is defined by analogy, but this time $z\in\mathcal{D}_-$).

Equalities (2) show that the condition $\|q\|=\sup_{x\in S}|q(x)|\le 1$ is necessary for the function g to take on only non-negative values. On the other hand, this condition is sufficient for that, as the following inequalities show (see Example 1.4.D)

$$(1-\|q\|)g_0(t,x,y)\le g(t,x,y)\le (1+\|q\|)g_0(t,x,y),\ t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d.$$

As a function of the arguments $(t,x)\in(0,+\infty)\times\mathbb{R}^d$ (for fixed $y\in\mathbb{R}^d$), the second item on the right-hand side of (1) can be considered as a single-layer potential. Hence, applying the main theorem of Section 5.2, we get the relations

$$\frac{\partial g(t,z,y)}{\partial\nu_z}\Big|_{z=x\pm}=(1\mp q(x))\frac{\partial g_0(t,x,y)}{\partial\nu_x} \tag{3}$$

valid for all $t>0, x\in S$ and $y\in\mathbb{R}^d$.

It was an exercise for the reader to verify that the function g defined by (1) satisfies the Kolmogorov–Chapman equation and also the continuity condition of Lesson 1. Having done that exercise, the reader must be sure that the function (1) determines a diffusion process in $\mathbb{R}^d$ (in some generalized sense, see Example 2.4.C) with its diffusion operator being an identity operator in $\mathbb{R}^d$ and its drift vector given by the function $(\nu q(x)\delta_S(x))_{x\in\mathbb{R}^d}$. Our aim now is to perturb this process by the S-valued vector field $(\alpha(x)\delta_S(x))_{x\in\mathbb{R}^d}$.

To do this, we need some information about the partial derivatives of the function (1) as a function of the argument $x\in\mathbb{R}^d$. Relations (3) characterize such a derivative in the direction ν. The following assertion contains some information about the corresponding derivatives in directions lying in S (under some additional assumptions on the function q).

Recall that an $\mathbb{R}^m$-valued ($m\ge 1$) function $(f(x))_{x\in S}$ is called Hölder continuous with the exponent $\lambda\in(0,1]$ if

$$\sup_{\substack{x\in S, y\in S\\ x\ne y}}\frac{|f(x)-f(y)|}{|x-y|^\lambda}<\infty. \tag{4}$$

One more group of reminders: for $x\in\mathbb{R}^d$, we make use of the designation $\widetilde{x}$ for the orthogonal projection of x on S (see Example 1.4.D); for given real numbers a and b, the minimal one of them is denoted by $a\wedge b$.

Let β be an arbitrary unit vector in S. The derivative in the direction β of the function $(g(t,x,y))_{x\in\mathbb{R}^d}$ (for fixed $t>0$ and $y\in\mathbb{R}^d$) will be denoted by

$$\frac{\partial g(t,x,y)}{\partial\beta_x}=(\beta,\nabla_x g(t,x,y)).$$

Lemma 1. *Assume that the function $(q(x))_{x\in S}$ with its values in $[-1,1]$ is Hölder continuous with the exponent $\lambda\in(0,1]$. Then the function g defined by (1) as a function of $x\in\mathbb{R}^d$ is differentiable in any direction $\beta\in S$ ($|\beta|=1$) and the derivative $\frac{\partial g(t,x,y)}{\partial\beta_x}$ ($t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d$) satisfies the following relations:*

(i)

$$\frac{\partial g(t,x,y)}{\partial\beta_x}=\frac{\partial g_0(t,x,y)}{\partial\beta_x}+q(\widetilde{x})\,\mathrm{sign}(y,\nu)+\frac{\partial g_0(t,\widetilde{x},\widetilde{y})}{\partial\beta_x}\exp\Big\{-\frac{(|(x,\nu)|+|(y,\nu)|)^2}{2t}\Big\}+$$

$$+\int_0^t d\tau\int_S\frac{\partial g_0(t-\tau,x,\zeta)}{\partial\beta_x}\frac{\partial g_0(\tau,\zeta,y)}{\partial\nu_\zeta}(q(\zeta)-q(\widetilde{x}))d\sigma_\zeta,\ t>0, x\in\mathbb{R}^d, y\in\mathbb{R}^d;$$

(ii) for any $T \in (0,+\infty)$, there exist constants $C > 0$ and $\mu > 0$ (they may be chosen independent of $\beta \in S, |\beta| = 1$) such that

$$\Big|\frac{\partial g(t,x,y)}{\partial \beta_x}\Big| \le Ct^{-\frac{d+1}{2}} \exp\Big\{-\mu\frac{|x-y|^2}{t}\Big\}, t \in (0,T], x \in \mathbb{R}^d, y \in \mathbb{R}^d;$$

(iii) for any $T \in (0,+\infty)$, there exist some constants $C > 0$ and $\mu > 0$ such that the inequality

$$\Big|\frac{\partial g(t,x,y)}{\partial \beta_x} - \frac{\partial g(t,z,y)}{\partial \beta_z}\Big| \le C|x-z|^\lambda t^{-\frac{d+1+\lambda}{2}} \exp\Big\{-\frac{\mu}{t}[(|x-\widetilde{y}| \wedge [z-\widetilde{y}])^2 + (y,\nu)^2]\Big\}$$

holds true for all $t \in (0,T], x \in S, y \in \mathbb{R}^d, z \in S, \beta \in S$ ($|\beta| = 1$).

Exercise 6.2.A. Prove all the statements of Lemma 1.

Hints. The integrals in the equality (i) are well-defined since $(q(x))_{x\in S}$ is Hölder continuous; the validity of that equality can be easily verified. The estimate (ii) is a simple consequence of (i) and the inequality in Theorem 1 from Lesson 3. Proving the estimate (iii) is not a difficult thing for the reader having acquired knowledge of the parametrix method for constructing fundamental solutions to parabolic equations (that is, in proving the theorem just cited).

Together with the function $(q(x))_{x\in S}$ taking on its values from the interval $[-1,1]$, we are given by an S-valued bounded function $(\alpha(x))_{x\in S}$. In what follows, they are both supposed to be Hölder continuous with the exponent $\lambda \in (0,1)$. As mentioned above, our aim is to construct a diffusion process in $\mathbb{R}^d$ with its diffusion operator being an identity operator in $\mathbb{R}^d$ and its drift vector given by the function $(\alpha(x) + q(x)\nu)\delta_S(x), x \in \mathbb{R}^d$.

Denote by $G(t,x,y)$ for $t > 0,\ x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ transition probability density of the process desired. In order to construct the function G, we make use of the first one of equations (5) from the previous lesson. In that equation, the function g is now defined by (1) (remind that for fixed $t > 0$ and $x \in \mathbb{R}^d$, the function $(g(t,x,y))_{y\in S}$ coincides with $(g_0(t,x,y))_{y\in S}$) and the function $(a(x))_{x\in\mathbb{R}^d}$ is now given by $(\alpha(x)\delta_S(x))_{x\in\mathbb{R}^d}$. Introducing the notation $V(\tau,z,y) = (\alpha(z), \nabla_z G(\tau,z,y))$ for $\tau > 0, z \in S$ and $y \in \mathbb{R}^d$ (this is an unknown function as well as the function G), we arrive at the following integro-differential equation

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau \int_S g_0(t-\tau,x,z)V(\tau,z,y)d\sigma_z, \qquad (5)$$

where $t > 0, x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$.

Suppose that the function $(V(\tau,z,y))_{z\in S}$ for fixed $\tau > 0$ and $y \in \mathbb{R}^d$ is Hölder continuous and put

$$\psi(t,x,y) = (\alpha(x), \nabla_x g(t,x,y)), K(t,x,y) = (\alpha(x), \nabla_x g_0(t,x,y))$$

for $t > 0, x \in S$ and $y \in \mathbb{R}^d$. Then equation (5) implies the following integral equation for the function V

$$V(t,x,y) = \psi(t,x,y) + \int_0^t d\tau \int_S K(t-\tau,x,z)V(\tau,z,y)d\sigma_z, \qquad (6)$$

where $t > 0, x \in S$ and $y \in \mathbb{R}^d$. Having solved this equation, one should then substitute the solution into equation (5), in order to obtain transition probability density of the process desired.

If we are not interested in the question whether that transition probability density does exist, we can simplify our problem. Let us multiply both sides of (5) and (6) by an arbitrary Borel measurable bounded function $(\varphi(y))_{y\in\mathbb{R}^d}$ and integrate them over $y \in \mathbb{R}^d$. Putting

$$T_t\varphi(x) = \int_{\mathbb{R}^d} G(t,x,y)\varphi(y)dy,\ (t,x) \in (0,+\infty) \times \mathbb{R}^d,$$

$$V_\varphi(t,x) = \int_{\mathbb{R}^d} V(t,x,y)\varphi(y)dy,\ (t,x)\in(0,+\infty)\times S,$$

$$\psi_\varphi(t,x) = \int_{\mathbb{R}^d} \psi(t,x,y)\varphi(y)dy,\ (t,x)\in(0,+\infty)\times S,$$

we get the following integral equation ($t>0, x\in S$)

$$V_\varphi(t,x) = \psi_\varphi(t,x) + \int_0^t d\tau \int_S K(t-\tau,x,z)V_\varphi(\tau,z)d\sigma_z \tag{7}$$

and the relation for $T_t\varphi(x), t>0, x\in\mathbb{R}^d$

$$T_t\varphi(x) = \int_{\mathbb{R}^d} g(t,x,y)\varphi(y)dy + \int_0^t d\tau \int_S g_0(t-\tau,x,z)V_\varphi(\tau,z)d\sigma_z. \tag{8}$$

The problem now consists in solving equation (7). After that, one should show that relation (8) determines the process desired.

6.3. Regularizing and solving equation (7). As in the previous lesson, we use the notation $\mathbb{B}(\mathbb{R}^d)$ for the Banach space of all real-valued bounded Borel measurable functions $(\varphi(x))_{x\in\mathbb{R}^d}$ with the norm $\|\varphi\| = \sup_{x\in\mathbb{R}^d}|\varphi(x)|$ and $\mathbb{C}(\mathbb{R}^d)$ for the closed subspace of $\mathbb{B}(\mathbb{R}^d)$ consisting of all continuous functions.

We are going to deal with equation (7) which is a Volterra integral equation of the second kind. It is determined by the function $\psi_\varphi(t,x),\ (t,x)\in(0,+\infty)\times S, \varphi\in\mathbb{B}(\mathbb{R}^d)$, and the kernel $K(t,x,y),\ (t,x)\in(0,+\infty)\times S, y\in\mathbb{R}^d$. According to Lemma 1, the function ψ_φ processes the following properties:

a) for any $T>0$, there exists a constant $C>0$ such that $|\psi_\varphi(t,x)|\le C\|\varphi\|t^{-1/2}$ for all $(t,x)\in(0,T]\times S$ and $\varphi\in\mathbb{B}(\mathbb{R}^d)$;

b) for any $T>0$. there exists a constant $C>0$ such that $|\psi_\varphi(t,x)-\psi_\varphi(t,z)|\le C\|\varphi\|t^{-(1+\lambda)/2}|x-z|^\lambda$ for all $t>0, x\in S, z\in S$ and $\varphi\in\mathbb{B}(\mathbb{R}^d)$.

As for the kernel K, the following estimate is a simple consequence of the inequality (ii) of Lemma 1:

for any $T>0$, there exist constants $C>0$ and $\mu>0$ such that

$$|K(t,x,z)|\le Ct^{-\frac{d+1}{2}}\exp\{-\mu|x-z|^2/t\},\ t\in(0,T], x\in S, z\in S.$$

Notice that this estimate does not allow one to conclude that this kernel has a weak singularity (the integral operator in (7) acts in the space of functions defined on $(0,+\infty)\times S$). Therefore, we cannot apply the method of successive approximation immediately to equation (7): it must be first regularized.

With that end in view, introduce into considerations an integro-differential operator $\mathcal{E}$ acting on a real-valued function $f(t,x)$ ($(t,x)\in(0,+\infty)\times S$) in accordance with the rule

$$\mathcal{E}f(t,x) = 2\Big\{\frac{\partial}{\partial t}\int_0^t d\tau \int_S f(\tau,y)\Big[\int_0^\infty h_0(t-\tau,\zeta)h(\hat{t}-\tau,x+\zeta\alpha(y),y)d\zeta\Big]d\sigma_y\Big\}\Big|_{\hat{t}=t},$$

where $h_0(t,\zeta) = (2\pi t)^{-1/2}\exp\{-\zeta^2/2t\}$ for $t>0$ and $\zeta\in\mathbb{R}^1$ and $h(t,x,y) = (2\pi t)^{-(d-1)/2}\exp\{-|y-x|^2/2t\}$ for $t>0, x\in S$ and $y\in S$.

Applying the operator $\mathcal{E}$ to equation (7) leads us to the following equation ($t>0$, $x\in S$)

$$V_\varphi(t,x) = \widetilde{\psi}_\varphi(t,x) + \int_0^t d\tau \int_S \widetilde{K}(t-\tau,x,y)V_\varphi(\tau,y)d\sigma_y, \tag{9}$$

where $\widetilde{\psi}_\varphi(t,x)$ for $t>0$ and $x\in S$ is given by the equality

$$\widetilde{\psi}_\varphi(t,x) = \psi_\varphi(t,x) + 2\int_0^t d\tau \int_S \psi_\varphi(\tau,\xi)d\sigma_\xi \int_0^\infty \frac{\partial h_0(t-\tau,\zeta)}{\partial t}\cdot \tag{10}$$

$$\cdot[h(t-\tau, x+\alpha(\xi)\zeta, \xi) - h(t-\tau, x+\alpha(x)\zeta, \xi)]d\zeta + 2\int_0^t d\tau \int_S [\psi_\varphi(\tau,\xi) -$$

$$-\psi_\varphi(\tau, x)]d\sigma_\xi \int_0^\infty \frac{\partial h_0(t-\tau,\zeta)}{\partial t} h(t-\tau, x+\alpha(x)\zeta, y)d\zeta$$

and $\widetilde{K}(t-\tau,x,y)$ for $0<\tau<t, x\in S$ and $y\in S$ is defined by the formula

$$\widetilde{K}(t-\tau,x,y) =$$

$$= 2\int_\tau^t ds \int_0^\infty \frac{\partial h_0}{\partial t}(t-s,\zeta)d\zeta \int_S [h(t-s, x+\alpha(\xi)\zeta,\xi)(\alpha(\xi), \nabla_\xi g_0(s-\tau,\xi,y) - \quad (11)$$

$$-h(t-s,x+\alpha(y)\zeta,\xi)(\alpha(y),\nabla_\xi g_0(s-\tau,\xi,y))]d\sigma_\xi + (\alpha(x)-\alpha(y), \nabla_x g_0(t-\tau,x,y)).$$

The following two exercises must be not difficult to those readers who have already mastered the parametrix method for constructing fundamental solutions to parabolic equations.

Exercise 6.3.A. Making use of representation (10), verify that the function $\widetilde{\psi}_\varphi$ possesses the properties a) and b) above as well as the function ψ_φ does.

Exercise 6.3.B. Using formula (11), prove that for any $T>0$ there exist some constants $C>0$ and $\mu>0$ such that the estimate

$$|\widetilde{K}(t-\tau,x,y)| \le C(t-\tau)^{-(d+1-\lambda)/2}\exp\{-\mu|y-x|^2/(t-\tau)\} \quad (12)$$

holds true for all $0\le\tau<t\le T,\ x\in S$ and $y\in S$.

Inequality (12) shows that the kernel $\widetilde{K}$ has a weak singularity. Taking into account additionally that the function $\widetilde{\psi}_\varphi$ possesses the property a) (see Example 6.3.A), we arrive at the conclusion that the method of successive approximations is applicable to equation (9). As a result, we have the following statement.

Lemma 2. *Let the function $(q(x))_{x\in S}$ with its values in the interval $[-1,1]$ and the S-valued bounded function $(\alpha(x))_{x\in S}$ be Hölder continuous with the exponent $\lambda\in(0,1)$. Then for any $\varphi\in\mathbb{B}(\mathbb{R}^d)$, there exists a solution $V_\varphi(t,x), (t,x)\in(0,+\infty)\times S$, of equation (9) such that: α) V_φ is continuous with respect to its arguments and for any $T<\infty$ it satisfies the inequality*

$$|V_\varphi(t,x)| \le C\|\varphi\|t^{-1/2} \quad (13)$$

for all $(t,x)\in(0,T]\times S$ with some constant $C>0$; β) there is only one solution of equation (9) possessing the property α); γ) if a sequence $\{\varphi_n, n=1,2,\ldots\}$ of real-valued Borel measurable functions $(\varphi_n(x))_{x\in\mathbb{R}^d}$ is such that $\lim_{n\to\infty}\varphi_n(x)=\varphi(x)$ for all $x\in\mathbb{R}^d$ and $\sup_{n\ge1}\|\varphi_n\|<\infty$, then $\lim_{n\to\infty}V_{\varphi_n}(t,x)=V_\varphi(t,x)$ for all $t>0$ and $x\in S$; δ) let $\varphi\in\mathbb{B}(\mathbb{R}^d)$ be such a function that its gradient $(\nabla\varphi(x))_{x\in\mathbb{R}^d}$ is bounded and Hölder continuous with the exponent $\lambda\in(0,1)$, then the restriction of V_φ on the domain $[0,T]\times S$ is Hölder continuous in the argument $t\in[0,T]$ and the argument $x\in S$ with the exponents $\lambda/2$ and λ, respectively.

Exercise 6.3.C. Prove all the assertions of Lemma 2.

Hints. Assertions $\alpha)-\gamma)$ easily follow from the construction of successive approximations for the solution V_φ of equation (9). The statement δ) is a consequence of Lemmas 1 and 2 in [17].

A question now arises, how does a solution of equation (9) is related to equation (7). The proof of the following statement can be found in [17].

Lemma 3. *The equations (7) and (9) are equivalent in the following sense: every solution of (9) is also a solution of (7) and vice versa.*

We have thus had a solution V_φ of equation (7) for an arbitrary $\varphi\in\mathbb{B}(\mathbb{R}^d)$. Substituting it into relation (8), we obtain a family of linear operators $(T_t)_{t>0}$ acting in the space $\mathbb{B}(\mathbb{R}^d)$. Estimate (13) implies the boundedness of those operators. Our nearest aim is to show that this family of operators determines a diffusion process in $\mathbb{R}^d$ (in the same sense as in Lesson 5), the existence of which was declared in Introduction to this lesson.

6.4. Constructing the process desired. Taking into account assertion γ) of Lemma 2, we can assert that $\lim_{n\to\infty} T_t\varphi_n(x) = T_t\varphi(x)$ for all $t > 0, x \in \mathbb{R}^d$ and any sequence $\{(\varphi_n(x))_{x\in\mathbb{R}^d}, n = 1, 2, \dots\}$ of functions from $\mathbb{B}(\mathbb{R}^d)$ such that $\sup_{n\geq 1} \|\varphi_n\| < +\infty$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ at each $x \in \mathbb{R}^d$. Introduce the notation $u(t, x, \varphi) = T_t\varphi(x)$ for $t > 0, x \in \mathbb{R}^d$ and $\varphi \in \mathbb{B}(\mathbb{R}^d)$ and notice that this function is the sum of two potentials: one of them is a volumetric heat potential and the other one is a single-layer potential associated with transition probability density g. Well-known properties of those potentials allow one to verify the following properties of the functions $u(t, x, \varphi), t > 0, x \in \mathbb{R}^d$:

1) it is a continuous function of the arguments $t > 0$ and $x \in \mathbb{R}^d$;

2) it satisfies the heat equation

$$\frac{\partial u(t, x, \varphi)}{\partial t} = \frac{1}{2}\Delta u(t, x, \varphi)$$

in the region $t > 0,\ x \in \mathcal{D}_+ \cup \mathcal{D}_-$;

3) for each $\varphi \in \mathbb{C}(\mathbb{R}^d)$, it satisfies the initial condition

$$u(0+, x, \varphi) = \varphi(x)$$

at any point $x \in \mathbb{R}^d$;

4) it satisfies the boundary condition

$$\frac{\partial u(t, x, \varphi)}{\partial \alpha(x)} + \frac{1 + q(x)}{2}\frac{\partial u(t, z, \varphi)}{\partial \nu}\Big|_{z=x+} - \frac{1 - q(x)}{2}\frac{\partial u(t, z, \varphi)}{\partial \nu}\Big|_{z=x-} = 0$$

for all $t > 0, x \in S$ and $\varphi \in \mathbb{B}(\mathbb{R}^d)$.

The uniqueness of a solution to this problem follows from the book [12] (see Chapter II, §5). As a consequence of this, we have the relation

$$u(s + t, x, \varphi) = u(t, x, u(s, \cdot, \varphi))$$

valid for all $s > 0, t > 0, x \in \mathbb{R}^d$ and $\varphi \in \mathbb{B}(\mathbb{R}^d)$. In addition, $u(t, x, \varphi_0) \equiv 1$ for $t > 0$, where $\varphi_0(x) \equiv 1$. Finally, if $\varphi \in \mathbb{B}(\mathbb{R}^d)$ is such that $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^d$, then $u(t, x, \varphi) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}^d$.

All these properties lead us to the conclusion that there exists transition probability $P(t, x, dy)$ in $(\mathbb{R}^d, \mathcal{B})$ such that

$$u(t, x, \varphi) = \int_{\mathbb{R}^d} \varphi(y)P(t, x, dy),\ t > 0, x \in \mathbb{R}^d,\ \varphi \in \mathbb{B}(\mathbb{R}^d).$$

The fact that there is an integral representation for the function $u(t, x, \varphi), t > 0, x \in \mathbb{R}^d$ and $\varphi \in \mathbb{B}(\mathbb{R}^d)$ (see (7) and (8)) must help to the reader to cope with the following exercises (compare with Exercises 5.5.D and 5.5.E).

Exercise 6.4.A. Make sure that for any $T > 0$, there exists a constant $C > 0$ such that the inequality

$$\int_{\mathbb{R}^d} |y - x|^4 P(t, x, dy) \leq Ct^2$$

holds true for all $t \in (0, T]$ and $x \in \mathbb{R}^d$.

Exercise 6.4.B. Verify that for any compactly supported function $\varphi \in \mathbb{C}(\mathbb{R}^d)$ and any $\theta \in \mathbb{R}^d$, the following relations (remind that $N(x) = \alpha(x) + q(x)\nu$ for $x \in S$)

$$\lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} \varphi(x)\Big[\int_{\mathbb{R}^d} (y - x, \theta)P(t, x, dy)\Big]dx = \int_S (N(x), \theta)\varphi(x)d\sigma_x,$$

$$\lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} \varphi(x)\Big[\int_{\mathbb{R}^d} (y - x, \theta)^2 P(t, x, dy)\Big]dx = |\theta|^2 \int_{\mathbb{R}^d} \varphi(x)dx$$

are fulfilled.

As a summation of the considerations of this lesson, we have the following result.

Theorem. *Let a bounded* S*-valued function* $(\alpha(x))_{x\in S}$ *and a function* $(q(x))_{x\in S}$ *with its values in the interval* $[-1, 1]$ *be given such that they both are Hölder continuous with some exponent* $\lambda \in (0, 1)$*. Then there exists a continuous Markov process* $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ *in* $\mathbb{R}^d$ *being a diffusion one (in a generalized sense) with its diffusion operator given by an identity operator in* $\mathbb{R}^d$ *and its drift vector given by the function* $((\alpha(x) + q(x)\nu)\delta_S(x))_{x\in\mathbb{R}^d}$.

Exercise 6.4.C. Try to construct the generalized diffusion process of this lesson making use of the second one of equations (5) from the previous lesson.

6.5. Comments and references. This lesson contains the results (with some modifications) of the paper [17].

References

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Chestochowa University of Technology, Chestochowa, Poland
E-mail address: bohdan.kopytko@im.pcz.pl

Institute of Mathematics of National Academy of Science of Ukraine, Kyiv, Ukraine
E-mail address: portenko@imath.kiev.ua